

Strong Solution of Backward Stochastic Partial Differential Equations in C^2 Domains*

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Abstract

This paper is concerned with the strong solution to the Cauchy-Dirichlet problem for backward stochastic partial differential equations of parabolic type. Existence and uniqueness theorems are obtained, due to an application of the continuation method under fairly weak conditions on variable coefficients and C^2 domains. The problem is also considered in weighted Sobolev spaces which allow the derivatives of the solutions to blow up near the boundary. As applications, a comparison theorem is obtained and the semi-linear equation is discussed in the C^2 domain.

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1 Introduction

In this paper we consider the Cauchy-Dirichlet problem for backward stochastic partial different equations (BSPDEs, for short) either in the non-divergence form:

$$\begin{aligned} dp(t, x) = & - [a^{ij}(t, x)p_{x^i x^j}(t, x) + b^i(t, x)p_{x^i}(t, x) - c(t, x)p(t, x) \\ & + \sigma^{ik}(t, x)q_{x^i}^k(t, x) + \nu^k(t, x)q^k(t, x) + F(t, x)]dt \\ & + q^k(t, x)dW_t^k, \quad (t, x) \in [0, T) \times \mathcal{D} \end{aligned} \quad (1.1)$$

or in the divergence form:

$$dp = -[(a^{ij}p_{x^j} + \sigma^{ik}q^k)_{x^i} + b^i p_{x^i} - cp + \nu^k q^k + F]dt + q^k dW_t^k \quad (1.2)$$

with the terminal-boundary condition:

$$\begin{cases} p(t, x) = 0, & t \in [0, T], \quad x \in \partial\mathcal{D}, \\ p(T, x) = \phi(x), & x \in \mathcal{D}. \end{cases} \quad (1.3)$$

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Here \mathcal{D} is a domain of the d -dimensional Euclidean space, and $W \triangleq \{W_t^k; t \geq 0\}$ is a d_1 -dimensional standard Wiener process, whose natural augmented filtration is denoted by $\{\mathcal{F}_t\}_{t \geq 0}$. The coefficients a, b, c, σ, ν and the free term F and the terminal condition ϕ are all random fields. An adapted solution of equation (1.1) or (1.2) is an $\mathcal{P} \times B(\mathcal{D})$ -measurable function pair (p, q) , which satisfies, in addition to (1.3), equations (1.1) or (1.2) under some appropriate sense, where \mathcal{P} is the predictable σ -algebra generated by $\{\mathcal{F}_t\}_{t \geq 0}$.

BSPDEs, which are a mathematically natural extension of backward SDEs (see e.g. [7, 17]), arise in many applications of probability theory and stochastic processes, for instance, in the optimal control of SDEs with incomplete information or more generally of stochastic parabolic PDEs, as adjoint equations (usually in the form of (1.2)) of Duncan-Mortensen-Zakai filtering equations (see e.g. [2, 16, 19, 20, 25]) to formulate the stochastic maximum principle for the optimal control, and in the formulation of the stochastic Feynman-Kac formula (see e.g. [14]) in mathematical finance. A class of fully nonlinear BSPDEs, the so-called backward stochastic Hamilton-Jacobi-Bellman equations, appears naturally in the dynamic programming theory of controlled non-Markovian processes (see [8, 18]). For more aspects of BSPDEs, we refer to e.g. [1, 21, 22, 23].

Equation (1.2) is usually understood in the weak sense (see Definition 2.1 (ii) in Section 2). When the coefficients a and σ are differentiable in x , equation (1.1) can be written into the divergence form (1.2). The existence and uniqueness of the weak solution of equation (1.2) in the whole space \mathbb{R}^d follows from that of backward stochastic evolution equations in Hilbert spaces (see e.g. [5, Prop. 3.2]). However, weak solutions have low regularity, which find difficulty in many applications. Strong solutions and even classical solutions are required in many occasions. In the case of $\mathcal{D} = \mathbb{R}^d$, the theory of strong solutions on BSPDEs is now fairly complete. For instance, a W_2^n -theory of the Cauchy problem for BSPDEs can be found in [2, 5, 11, 15, 24]. On the contrary, there are very few discussions on the Cauchy-Dirichlet problem for BSPDEs. Here we could mention only two works. A special form of equation (1.2) with Dirichlet conditions is studied in [23] by the method of semigroups, in the context that the coefficients are independent of (ω, t) . The other is our previous work [6], where the equations are solved in weighted Sobolev spaces.

A main difficulty in strong solution of the Cauchy-Dirichlet problem for BSPDEs (and SPDEs) is to estimate the second order partial derivatives of the solution. In the theory of deterministic PDEs, it is solved with the help of the estimate of the derivative in t , which makes any sense in general neither for BSPDEs nor for SPDEs. For SPDEs, Flandoli [10] establishes some regularity under additional compatibility conditions, and Krylov [12] studies the equations in weighted Sobolev spaces allowing the derivatives of the solutions to blow up near the boundary of \mathcal{D} . Note that there is an essential difference between SPDEs and BSPDEs: the noises in the former are exogenous and play an active role, while in the latter they are governed by the randomness of the coefficients and the terminal condition and thus they are endogenous, coming from martingale representations. The regularity of BSPDEs turns out to be more like deterministic PDEs than that of SPDEs.

In this paper, we prove the existence and uniqueness of the strong solution (see Definition 2.1 (i) in Section 2) of equation (1.1) without involving any additional compatibility condition nor any weighting functions. Our approach is based on the method of the odd reflection and some classical techniques from the theory of deterministic PDEs. Our assumptions on the coefficients are rather natural since they include the rather general

deterministic PDEs where the leading coefficients are not necessarily differentiable in x . Unfortunately, in contrast to deterministic parabolic PDEs (see e.g. Theorem 7.1.6 in Evans [9]), further regularity for BSPDEs seems to be hopeless since the unknown random fields are not expected to be differentiable with respect to t as in the classical sense. However, we can consider the equations in weighted Sobolev spaces which allow the derivatives of the solutions to blow up near the boundary. Starting from the existence and uniqueness of the strong solution, we prove a slightly different version of our previous work [6], and obtain the interior regularity for equation (1.1). In the last part of our paper, we prove a comparison theorem for the strong solution of equation (1.1), and we also discuss a class of semi-linear BSPDEs in C^2 domains.

The rest of the paper is organized as follows. In Section 2, we introduce some notations and preliminary results. In Section 3, we state our main existence and uniqueness result in Theorem 3.1, on the basis of which we study the equations in weighted Sobolev spaces. Section 4 is devoted to the proof of Theorem 3.1, which is divided into two subsections. Finally, in Section 5, we prove a comparison theorem, and discuss semi-linear BSPDEs in C^2 domains.

2 Preliminaries

2.1 Notations

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$ be a complete filtered probability space on which is defined a d_1 -dimensional standard Wiener process $W = \{W_t; t \geq 0\}$ such that $\{\mathcal{F}_t\}_{t \geq 0}$ is the natural filtration generated by W , augmented by all the P -null sets in \mathcal{F} . Fix a positive number T . Denote by \mathcal{P} the σ -algebra of predictable sets on $\Omega \times [0, T)$ associated with $\{\mathcal{F}_t\}_{t \geq 0}$.

Let \mathcal{D} be a domain in \mathbb{R}^d with boundary of class C^2 .

For the sake of convenience, we denote

$$D_i u = u_{x^i}, \quad D_{ij} u = u_{x^i x^j}, \quad Du = u_x = (D_1 u, \dots, D_d u), \quad D^2 u = (D_{ij} u)_{d \times d},$$

and for any multi-index $\alpha = (\alpha_1, \dots, \alpha_d)$

$$D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2} \dots D_d^{\alpha_d}, \quad |\alpha| = \alpha_1 + \dots + \alpha_d.$$

For any two multi-indices $\alpha = (\alpha_1, \dots, \alpha_d)$ and $\beta = (\beta_1, \dots, \beta_d)$, we define

$$\alpha + \beta = (\alpha_1 + \beta_1, \dots, \alpha_d + \beta_d).$$

We shall also use the summation convention.

Now we introduce some function spaces. For any integer $k \geq 0$, we denote by $C^k(\mathcal{D})$ (or $C^k(\overline{\mathcal{D}})$) the set of functions having all derivatives up to order k continuous in \mathcal{D} (or $\overline{\mathcal{D}}$); by $C_b^k(\mathcal{D})$ (or $C_b^k(\overline{\mathcal{D}})$) the set of those functions in $C^k(\mathcal{D})$ (or $C^k(\overline{\mathcal{D}})$) whose partial derivatives up to order k are uniformly bounded in \mathcal{D} (or $\overline{\mathcal{D}}$).

For a given Banach space \mathcal{B} , denote by $L^2(\Omega \times (0, T), \mathcal{P}, \mathcal{B})$ the space of all \mathcal{B} -valued predictable process $X : \Omega \times [0, T] \rightarrow \mathcal{B}$ such that $E \int_0^T \|X(t)\|_{\mathcal{B}}^2 dt < \infty$.

Let $H^m(\mathcal{D})$ be the Sobolev space $W^{m,2}(\mathcal{D})$ and $H_0^m(\mathcal{D}) = W_0^{m,2}(\mathcal{D})$. Denote

$$\begin{aligned}\mathbb{H}^0(\mathcal{D}) &= L^2(\Omega \times (0, T), \mathcal{P}, L^2(\mathcal{D})), \\ \mathbb{H}^m(\mathcal{D}) &= L^2(\Omega \times (0, T), \mathcal{P}, H^m(\mathcal{D})), \quad m = -1, 1, 2, \dots, \\ \mathbb{H}_0^n(\mathcal{D}) &= L^2(\Omega \times (0, T), \mathcal{P}, H_0^n(\mathcal{D})), \quad n = 1, 2, 3, \dots, \\ \mathbb{H}_0^1 \cap \mathbb{H}^2(\mathcal{D}) &= L^2(\Omega \times (0, T), \mathcal{P}, H_0^1(\mathcal{D}) \cap H^2(\mathcal{D})).\end{aligned}$$

Note that $\mathbb{H}_0^1(\mathbb{R}^d) = \mathbb{H}^1(\mathbb{R}^d)$, $\mathbb{H}_0^1 \cap \mathbb{H}^2(\mathbb{R}^d) = \mathbb{H}^2(\mathbb{R}^d)$. In addition, denote

$$\|\cdot\|_{0,\mathcal{D}} = \|\cdot\|_{L^2(\mathcal{D})}, \quad \|\cdot\|_{m,\mathcal{D}} = \|\cdot\|_{H^m(\mathcal{D})}, \quad m = -1, 1, 2, \dots$$

Moreover, for a function u defined on $\Omega \times [0, T] \times \mathcal{D}$, we denote

$$\|u\|_{m,\mathcal{D}}^2 = E \int_0^T \|u(t, \cdot)\|_{m,\mathcal{D}}^2 dt, \quad m = -1, 0, 1, 2, \dots$$

The same notations will be used for vector-valued and matrix-valued functions, and in the case we denote $|u|^2 = \sum_i |u^i|^2$ and $|u|^2 = \sum_{ij} |u^{ij}|^2$, respectively.

Furthermore, we define the two product spaces

$$\begin{aligned}\mathcal{H}^{2,1}(\mathcal{D}) &= (\mathbb{H}_0^1 \cap \mathbb{H}^2(\mathcal{D})) \times \mathbb{H}^1(\mathcal{D}; \mathbb{R}^{d_1}), \\ \mathcal{H}_0^{2,1}(\mathcal{D}) &= (\mathbb{H}_0^1 \cap \mathbb{H}^2(\mathcal{D})) \times \mathbb{H}_0^1(\mathcal{D}; \mathbb{R}^{d_1}),\end{aligned}\tag{2.1}$$

both being equipped with the norm

$$\|(u, v)\|_{\mathcal{H}^{2,1}(\mathcal{D})} = \left(\|u\|_{2,\mathcal{D}}^2 + \|v\|_{1,\mathcal{D}}^2 \right)^{1/2}.$$

It is clear that both $\mathcal{H}^{2,1}(\mathcal{D})$ and $\mathcal{H}_0^{2,1}(\mathcal{D})$ are Banach spaces.

2.2 An Itô formula

Let V and H be two separable Hilbert spaces such that V is densely embedded in H . We identify H with its dual space, and denote by V^* the dual of V . We have $V \subset H \subset V^*$. Denote by $\|\cdot\|_H$ the norms of H , by $\langle \cdot, \cdot \rangle_H$ the scalar product in H , and by $\langle \cdot, \cdot \rangle$ the duality product between V and V^* .

Consider three processes v, m , and v^* , defined on $\Omega \times [0, T]$, taking values in V, H and V^* , respectively. Let $v(\omega, t)$ be measurable with respect to (ω, t) and be \mathcal{F}_t -measurable with respect to ω for a.e. t . For any $\eta \in V$, the quantity $\langle \eta, v^*(\omega, t) \rangle$ is \mathcal{F}_t -measurable in ω for a.e. t and is measurable with respect to (ω, t) . Assume that $m(\omega, t)$ is strongly continuous in t and \mathcal{F}_t -measurable with respect to ω for any t , and that it is a local martingale. Let $\langle m \rangle$ be the increasing process in the Doob-Meyer Decomposition of the sub-martingale $\|m\|_H^2$ (see e.g. [13, Page 1240]).

Proceeding identically to the proof of Theorem 3.2 in Krylov and Rozovskii [13], we have the following result concerning Itô's formula, which is the backward version of [13, Theorem 3.2].

Lemma 2.1. *Let $\varphi \in L^2(\Omega, \mathcal{F}_T, H)$. Suppose that for every $\eta \in V$ and almost every $(\omega, t) \in \Omega \times [0, T]$, it holds that*

$$\langle \eta, v(t) \rangle_H = \langle \eta, \varphi \rangle_H + \int_t^T \langle \eta, v^*(s) \rangle ds + \langle \eta, m(T) - m(t) \rangle_H.$$

Then there exist a set $\Omega' \subset \Omega$ s.t. $P(\Omega') = 1$ and a function $h(t)$ with values in H such that

(a) *$h(t)$ is \mathcal{F}_t -measurable for any $t \in [0, T]$ and strongly continuous with respect to t for any ω , and $h(t) = v(t)$ (in the space H) a.e. $(\omega, t) \in \Omega \times [0, T]$, and $h(T) = \varphi$ for any $\omega \in \Omega'$;*

(b) *for any $\omega \in \Omega'$ and any $t \in [0, T]$,*

$$\|h(t)\|_H^2 = \|\varphi\|_H^2 + 2 \int_t^T \langle v(s), v^*(s) \rangle ds + 2 \int_t^T \langle h(s), dm(s) \rangle_H - \langle m \rangle_T + \langle m \rangle_t.$$

2.3 Notions of solutions to BSPDEs

Throughout this paper, we assume that the given functions

$$\begin{aligned} a &= (a^{ij}) : \Omega \times [0, T] \times \mathcal{D} \rightarrow S^d, & \sigma &= (\sigma^{ik}) : \Omega \times [0, T] \times \mathcal{D} \rightarrow \mathbb{R}^{d \times d_1}, \\ b &= (b^i) : \Omega \times [0, T] \times \mathcal{D} \rightarrow \mathbb{R}^d, & \nu &= (\nu^k) : \Omega \times [0, T] \times \mathcal{D} \rightarrow \mathbb{R}^{d_1}, \\ c, F &: \Omega \times [0, T] \times \mathcal{D} \rightarrow \mathbb{R} \end{aligned}$$

are all $\mathcal{P} \times B(\mathcal{D})$ -measurable (S^d is the set of real symmetric $d \times d$ matrices), and the function $\phi : \Omega \times \mathcal{D} \rightarrow \mathbb{R}$ is $\mathcal{F}_T \times B(\mathcal{D})$ -measurable.

Let us now turn to the notions of solutions to equations (1.1) and (1.2). Throughout this subsection it will be supposed that the coefficients of our equations, i.e., the functions a, b, c, σ and ν , are all bounded.

Definition 2.1. A pair of random fields $\{(p(\omega, t, x), q(\omega, t, x)); (\omega, t, x) \in \Omega \times [0, T] \times \mathbb{R}^d\}$ is called

(i) a strong solution of equation (1.1) with the terminal-boundary condition (1.3) if $(p, q) \in \mathcal{H}^{2,1}(\mathcal{D})$ and $p \in C([0, T], L^2(\mathcal{D}))$ (a.s.) such that for each $t \in [0, T]$ and a.s. $\omega \in \Omega$, it holds that

$$\begin{aligned} p(t, x) &= \phi(x) + \int_t^T [a^{ij}(s, x) D_{ij} p(s, x) + b^i(s, x) D_i p(s, x) - c(s, x) p(s, x) \\ &\quad + \sigma^{ik}(s, x) D_i q^k(s, x) + \nu^k(s, x) q^k(s, x) + F(s, x)] ds - \int_t^T q^k(s, x) dW_s^k \end{aligned} \quad (2.2)$$

for almost every $x \in \mathbb{R}^d$;

(ii) a weak solution of equation (1.2) with the terminal-boundary condition (1.3), if $(p, q) \in \mathbb{H}_0^1(\mathcal{D}) \times \mathbb{H}^0(\mathcal{D}; \mathbb{R}^{d_1})$ such that for every $\eta \in C_0^\infty(\mathcal{D})$ and almost every $(\omega, t) \in \Omega \times [0, T]$, it holds that

$$\begin{aligned} \int_{\mathcal{D}} p(t, \cdot) \eta dx &= \int_{\mathcal{D}} \phi \eta dx + \int_t^T \int_{\mathcal{D}} [(a^{ij} D_i p + \sigma^{jk} q^k) D_j \eta \\ &\quad + (b^i D_i p - cp + \nu^k q^k + F) \eta] dx dt - \int_t^T \int_{\mathcal{D}} q^k \eta dx dW_t^k. \end{aligned} \quad (2.3)$$

It follows from Lemma 2.1 that the first component of the weak solution of equation (1.2) has a continuous version in $L^2(\mathcal{D})$, i.e., $p \in C([0, T], L^2(\mathcal{D}))$ (a.s.). For the strong solution of equation (1.1) with the terminal-boundary condition (1.3), we have the following

Proposition 2.2. *Let $(p, q) \in \mathcal{H}^{2,1}(\mathcal{D})$. The following statements are equivalent:*

- (i) (p, q) is a strong solution of BSPDE (1.1) and (1.3);
- (ii) for every $\eta \in C_0^\infty(\mathcal{D})$ and a.e. $(\omega, t) \in \Omega \times [0, T]$, it holds that

$$\begin{aligned} \int_{\mathcal{D}} p(t, \cdot) \eta dx &= \int_{\mathcal{D}} \phi \eta dx + \int_t^T \int_{\mathcal{D}} [a^{ij} D_{ij} p + b^i D_i p - cp \\ &\quad + \sigma^{ik} D_i q^k + \nu^k q^k + F] \eta dx dt - \int_t^T \int_{\mathcal{D}} q^k \eta dx dW_t^k. \end{aligned} \quad (2.4)$$

Proof. It is clear that (i) \Rightarrow (ii). Now we prove (ii) \Rightarrow (i). It follows from Lemma 2.1 that $p \in C([0, T], L^2(\mathcal{D}))$ (a.s.). Then from the time continuity of the (stochastic) integrals, we know that equation (2.4) holds almost surely for every $\eta \in C_0^\infty(\mathcal{D})$ and all $t \in [0, T]$. On the other hand, since $(p, q) \in \mathcal{H}^{2,1}(\mathcal{D})$, both sides of equation (2.2) (as functions of x) belong to $L^2(\mathcal{D})$ a.s. for every t . Since $C_0^\infty(\mathcal{D})$ is dense in $L^2(\mathcal{D})$, we know that equation (2.2) holds in the space $L^2(\mathcal{D})$ for every t , which evidently implies (i). The proof is complete. \square

Remark 2.1. Note that the space of test functions $C_0^\infty(\mathcal{D})$ in Definition 2.1 and Proposition 2.2 can be replaced with $H_0^1(\mathcal{D})$.

If the coefficients a and σ are differentiable in x , then equation (1.1) can be written into the divergence form (1.2), which allows us to define the weak solution to (1.1). Then Proposition 2.2 indicates that a strong solution of (1.1) is a weak solution of (1.1).

The following lemma concerns the existence and uniqueness of the weak solution of equation (1.2), which is borrowed from Proposition 3.2 in [5]. On the other hand, it can be proved by the duality method as in Zhou [24] and Lemma 2.1.

Lemma 2.3. *Assume that $\kappa I + \sigma \sigma^* \leq 2a \leq \kappa^{-1} I$ for some constant $\kappa > 0$ and that the functions b^i, c and ν^k are bounded by K . Suppose that $F \in \mathbb{H}^{-1}(\mathcal{D})$ and $\phi \in L^2(\Omega, \mathcal{F}_T, L^2(\mathcal{D}))$. Then equation (1.2) with the terminal-boundary condition (1.3) has a unique weak solution (p, q) in the space $\mathbb{H}_0^1(\mathcal{D}) \times \mathbb{H}^0(\mathcal{D}; \mathbb{R}^{d_1})$ such that $p \in C([0, T], L^2(\mathcal{D}))$ (a.s.), and*

$$\| \| p \| \|_{1, \mathcal{D}}^2 + \| \| q \| \|_{0, \mathcal{D}}^2 + E \sup_{0 \leq t \leq T} \| p(t, \cdot) \|_{0, \mathcal{D}}^2 \leq C (\| \| F \| \|_{-1, \mathcal{D}}^2 + E \| \phi \|_{0, \mathcal{D}}^2), \quad (2.5)$$

where the constant $C = C(K, \kappa, T)$.

3 Existence, Uniqueness, and Regularity

In this section, we state our main results on the existence, uniqueness and regularity of the strong solution of equation (1.1) with the terminal-boundary condition (1.3).

3.1 Existence and uniqueness of strong solutions to BSPDEs

The weak solution of a deterministic parabolic PDE can be shown to belong to the space $L^2(0, T; H^2(\mathcal{D}))$ if the free term belongs to $L^2((0, T) \times \mathcal{D})$ and the initial data lies in $H_0^1(\mathcal{D})$ (see e.g. Evans [9]). Flandoli [10] formulates a counterpart for a SPDEs with additional compatibility conditions on the free term. In the following, we obtain a counterpart for a BSPDE, which, in a remarkable way, does not require any compatibility condition like a SPDE. The higher regularity of the (strong) solution allows us to weaken the assumptions on the leading coefficients a and σ so that they are not necessarily differentiable in x , where equation (1.1) is difficult to be written into the divergence form (1.2).

Fix some constants $K \in (1, \infty)$ and $\rho_0, \kappa \in (0, 1)$. Denote

$$B_+ = \{x \in \mathbb{R}^d : |x| < 1, x^1 > 0\}, \quad B_\rho(x) = \{y \in \mathbb{R}^d : |x - y| < \rho\}.$$

Assumption 3.1. For every $x \in \partial\mathcal{D}$ there exist a domain $U \subset B_{8K\rho_0}(x)$ containing the ball $B_{4\rho_0}(x)$ and a one-to-one map $\Phi : 2B_+ \rightarrow U \cap \mathcal{D}$ having the properties:

$$\begin{aligned} x &= \Phi(0), \quad \Phi(B_+) \supset B_{4\rho_0}(x) \cap \mathcal{D}, \quad \Phi(\partial B_+ \cap \{x^1 = 0\}) \subset \partial\mathcal{D}; \\ \kappa|\xi|^2 &\leq |(D\Phi)\xi|^2 \leq \kappa^{-1}|\xi|^2, \quad \forall \xi \in \mathbb{R}^d; \\ |D^\alpha \Phi| &\leq K \quad \text{for any multi-index } \alpha \text{ s.t. } |\alpha| \leq 2, \end{aligned}$$

where $D\Phi$ is the Jacobi matrix of Φ .

Note that in view of the Heine-Borel theorem, Assumption 3.1 is true if the domain \mathcal{D} is bounded and its boundary is C^2 .

Assumption 3.2. The *super-parabolicity* condition holds:

$$\kappa I + \sigma\sigma^* \leq 2a \leq \kappa^{-1}I, \quad \forall (\omega, t, x) \in \Omega \times [0, T] \times \mathcal{D}.$$

Assumption 3.3. There exists a function $\gamma : [0, \infty) \rightarrow [0, \infty)$ such that (i) γ is continuous and increasing, (ii) $\gamma(r) = 0$ if and only if $r = 0$, and (iii) for any $(\omega, t) \in \Omega \times [0, T]$ and any $x, y \in \mathcal{D}$,

$$|a(\omega, t, x) - a(\omega, t, y)| \leq \gamma(|x - y|), \quad |\sigma(\omega, t, x) - \sigma(\omega, t, y)| \leq \gamma(|x - y|). \quad (3.1)$$

We have the following existence and uniqueness theorem, whose proof will be given in the next section.

Theorem 3.1. *Let Assumptions 3.1, 3.2, and 3.3 be satisfied. Assume that the functions b^i, c , and ν^k are bounded by K . If $F \in \mathbb{H}^0(\mathcal{D})$ and $\phi \in L^2(\Omega, \mathcal{F}_T, H_0^1(\mathcal{D}))$, BSPDE (1.1) and (1.3) has a unique strong solution $(p, q) \in \mathcal{H}_0^{2,1}(\mathcal{D})$ such that*

$$p \in C([0, T], L^2(\mathcal{D})) \cap L^\infty([0, T], H^1(\mathcal{D})) \text{ (a.s.)}. \quad (3.2)$$

Moreover, we have the following estimate

$$\| \| p \| \|_{2,\mathcal{D}}^2 + \| \| q \| \|_{1,\mathcal{D}}^2 + E \sup_{0 \leq t \leq T} \| p(t, \cdot) \|_{1,\mathcal{D}}^2 \leq C (\| \| F \| \|_{0,\mathcal{D}}^2 + E \| \phi \|_{1,\mathcal{D}}^2), \quad (3.3)$$

where the constant C only depends on K, ρ_0, κ, T , and the function γ .

Remark 3.1. Since all constants C in this paper are independent of d_1 , all our results in this paper may be extended to the more general equation (1.1) where the d_1 -dimensional standard Wiener process is replaced with a cylindrical Wiener process.

3.2 Solution in weighted Sobolev spaces and regularity

Unfortunately, we could not establish any higher regularity for BSPDEs to correspond to the theory of deterministic parabolic PDEs, as given by Evans [9, Theorem 7.1.6], since the unknown functions are not expected to be differentiable with respect to t as in the deterministic sense. However, we can consider the equations in weighted Sobolev spaces allowing the derivatives of the solutions to blow up near the boundary, and furthermore obtain an interior regularity for BSPDE (1.1) and (1.3).

Let $\psi \in C_b^2(\overline{\mathcal{D}})$ be a nonnegative function such that $\psi(x) = 0$ for any $x \in \partial\mathcal{D}$. Fix a positive integer n .

Assumption 3.4. For any multi-index α such that $|\alpha| \leq n$, we have

$$\begin{aligned} \psi^{|\alpha|}(|D^\alpha a^{ij}| + |D^\alpha b^i| + |D^\alpha c| + |D^\alpha \sigma^{ik}| + |D^\alpha \nu^k|) &\leq K, \\ \psi^{|\alpha|} D^\alpha F &\in \mathbb{H}^0(\mathcal{D}), \quad \psi^{|\alpha|} D^\alpha \phi \in L^2(\Omega, \mathcal{F}_T, H_0^1(\mathcal{D})). \end{aligned}$$

Note that Assumption 3.4 implies the boundedness of the functions b, c and ν , but does not imply Assumption 3.3. We have the following

Theorem 3.2. *Let Assumptions 3.1, 3.2, 3.3, and 3.4 be satisfied. Then BSPDE (1.1) and (1.3) has a unique strong solution (p, q) such that for any multi-index α s.t. $|\alpha| \leq n$,*

$$\begin{aligned} (\psi^{|\alpha|} D^\alpha p, \psi^{|\alpha|} D^\alpha q) &\in \mathcal{H}_0^{2,1}(\mathcal{D}), \\ \psi^{|\alpha|} D^\alpha p &\in C([0, T], L^2(\mathcal{D})) \cap L^\infty([0, T], H^1(\mathcal{D})) \text{ (a.s.)}, \end{aligned} \tag{3.4}$$

and moreover

$$\begin{aligned} &\| \psi^{|\alpha|} D^\alpha p \|_{2,\mathcal{D}}^2 + \| \psi^{|\alpha|} D^\alpha q \|_{1,\mathcal{D}}^2 + E \sup_{0 \leq t \leq T} \| \psi^{|\alpha|} D^\alpha p(t, \cdot) \|_{1,\mathcal{D}}^2 \\ &\leq C \sum_{|\beta| \leq |\alpha|} \left(\| \psi^{|\beta|} D^\beta F \|_{0,\mathcal{D}}^2 + E \| \psi^{|\beta|} D^\beta \phi \|_{1,\mathcal{D}}^2 \right), \end{aligned} \tag{3.5}$$

where the constant C only depends on the norm of ψ in $C^2(\overline{\mathcal{D}})$, the parameters K, ρ_0, κ , and T , and the function γ .

We need the following lemma, which can be found in [12].

Lemma 3.3. *If both v and ψDv lie in $L^2(\mathcal{D})$, then $\psi v \in H_0^1(\mathcal{D})$.*

Proof. Define $\mathcal{K}_n = \{x \in \mathcal{D} : \text{dist}(x, \partial\mathcal{D}) \geq 4 \cdot 2^{-n}\}$. Take a nonnegative function $\zeta \in C_0^\infty(\mathbb{R}^d)$ such that $\text{supp} \zeta \subset B_1(0)$, $\int_{\mathbb{R}^d} \zeta = 1$. Define $\zeta_n(x) = 2^{nd} \zeta(2^n x)$ and $\eta_n = \zeta_n * 1_{\mathcal{K}_n}$. We have $\text{supp}(\eta_n) \subset \mathcal{K}_{n+1}$ and $\eta_n|_{\mathcal{K}_{n-1}} = 1$. Since $\psi \in C_b^2(\overline{\mathcal{D}})$, we have $|\psi(x)| \leq C \text{dist}(x, \partial\mathcal{D})$. It is not hard to show that $|\eta_n| \leq 1$, $|\psi D\eta_n| \leq C$, and $\eta_n \psi v \in H_0^1(\mathcal{D})$. Then we can get that both $\eta_n \psi v \rightarrow \psi v$ and $\eta_n D(\psi v) \rightarrow D(\psi v)$ strongly in $L^2(\mathcal{D})$ as $n \rightarrow \infty$, and moreover

$$\begin{aligned} \int_{\mathcal{D}} |D(\eta_n \psi v) - D(\psi v)|^2 &\leq 2 \int_{\mathcal{D}} |\psi D\eta_n|^2 |v|^2 + 2 \int_{\mathcal{D}} |\eta_n D(\psi v) - D(\psi v)|^2 \\ &\leq C \int_{\mathcal{D} \setminus \mathcal{K}_{n-1}} |v|^2 + 2 \int_{\mathcal{D}} |\eta_n D(\psi v) - D(\psi v)|^2 \\ &\rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Hence $\psi v \in H_0^1(\mathcal{D})$. □

Proof of Theorem 3.2. The proof consists of two steps. We suppose for the moment that $\psi \in C^{n+2}(\overline{\mathcal{D}})$, which will finally be dispensed with.

Step 1. We first prove that Theorem 3.2 is true for the case where the leading coefficients a and σ are differentiable in the space variable x with the gradients a_x and σ_x being bounded and thus equation (1.1) can be written into the divergence form (1.2).

We use the induction. Theorem 3.1 shows that Theorem 3.2 is true for the case of $n = 0$. Assume that it is true for the case of $n = m - 1$ ($m \geq 1$). It is sufficient for us to show that it is true for the case of $n = m$.

We assert that $(\psi^m D^\alpha p, \psi^m D^\alpha q) \in \mathbb{H}_0^1(\mathcal{D}) \times \mathbb{H}^0(\mathcal{D}; \mathbb{R}^{d_1})$ for any multi-index α s.t. $|\alpha| = m$. Indeed, we know from our assumption that for any multi-index β s.t. $|\beta| \leq m - 1$,

$$\psi^{m-1} D^\beta p \in \mathbb{H}^2(\mathcal{D}), \quad \psi^{m-1} D^\beta q \in \mathbb{H}^1(\mathcal{D}; \mathbb{R}^{d_1}).$$

Keeping in mind that $\psi \in C_b^2(\overline{\mathcal{D}})$, we can easily get by induction that

$$\psi^{m-1} D^\beta p_x \in \mathbb{H}^1(\mathcal{D}), \quad \psi^{m-1} D^\beta p_{xx} \in \mathbb{H}^0(\mathcal{D}), \quad \text{and } \psi^{m-1} D^\beta q_x \in \mathbb{H}^0(\mathcal{D}; \mathbb{R}^{d_1}). \quad (3.6)$$

Then we have

$$\psi^{m-1} D^\alpha p \in \mathbb{H}^1(\mathcal{D}) \subset \mathbb{H}^0(\mathcal{D}), \quad \psi D(\psi^{m-1} D^\alpha p) \in \mathbb{H}^0(\mathcal{D}), \quad \text{and } \psi^m D^\alpha q \in \mathbb{H}^0(\mathcal{D}; \mathbb{R}^{d_1}).$$

In view of Lemma 3.3, we have from the first two relations that $\psi^m D^\alpha p \in \mathbb{H}_0^1(\mathcal{D})$.

Take any $\eta \in C_0^\infty(\mathcal{D})$. Since $\psi \in C_b^{m+2}(\mathcal{D})$, we know that $D^\alpha(\psi^m \eta) \in H_0^1(\mathcal{D})$. From Proposition 2.2, we have

$$\begin{aligned} & \int_{\mathcal{D}} p(t, \cdot) D^\alpha(\psi^m \eta) dx - \int_{\mathcal{D}} \phi D^\alpha(\psi^m \eta) dx \\ &= \int_t^T \int_{\mathcal{D}} \left[a^{ij} D_{ij} p + b^i D_i p - c p + \sigma^{ik} D_i q^k + \nu^k q^k + F \right] D^\alpha(\psi^m \eta) dx dt \\ & \quad - \int_t^T \int_{\mathcal{D}} q^k D^\alpha(\psi^m \eta) dx dW_t^k, \quad \text{a.e. } (\omega, t) \in \Omega \times [0, T]. \end{aligned}$$

Using the integration by parts formula, we show that the function pair $(\psi^m D^\alpha p, \psi^m D^\alpha q) \in \mathbb{H}_0^1(\mathcal{D}) \times \mathbb{H}^0(\mathcal{D}; \mathbb{R}^{d_1})$ is a weak solution of the following BSPDE

$$\begin{cases} du = -[a^{ij} D_{ij} u + \sigma^{ik} D_i v^k + \tilde{F}] dt + v^k dW_t^k, \\ u(t, x) = 0, \quad t \in [0, T], \quad x \in \partial \mathcal{D}, \\ u(T, x) = (\psi^m D^\alpha \phi)(x), \quad x \in \mathcal{D} \end{cases} \quad (3.7)$$

with u and v being the unknown functions. Here

$$\begin{aligned} \tilde{F} &= \sum_{\beta+\gamma=\alpha, |\beta| \geq 1} \left[(\psi^{|\beta|} D^\beta a^{ij}) (\psi^{|\gamma|} D^\gamma p_{x^i x^j}) + (\psi^{|\beta|} D^\beta \sigma^{ik}) (\psi^{|\gamma|} D^\gamma q_{x^i}^k) \right] \\ &+ \sum_{\beta+\gamma=\alpha} \left[(\psi^{|\beta|} D^\beta b^i) (\psi^{|\gamma|} D^\gamma p_{x^i}) - (\psi^{|\beta|} D^\beta c) (\psi^{|\gamma|} D^\gamma p) \right] \\ &- 2m a^{ij} \psi^{m-1} D_i \psi D^\alpha p_{x^j} - m a^{ij} \psi^{m-1} D_{ij} \psi D^\alpha p - m(m-1) a^{ij} \psi^{m-2} D_i \psi D_j \psi D^\alpha p \\ &+ \sum_{\beta+\gamma=\alpha} (\psi^{|\beta|} D^\beta \nu^k) (\psi^{|\gamma|} D^\gamma q^k) - m \sigma^{ik} \psi^{m-1} D_i \psi D^\alpha q^k + \psi^m D^\alpha F. \end{aligned} \quad (3.8)$$

From (3.6) and Assumption 3.4, we see that $\tilde{F} \in \mathbb{H}^0(\mathcal{D})$. Moreover, from estimate (3.5) for $n = m - 1$ (as a consequence of the induction assumption), we have

$$\begin{aligned} \|\tilde{F}\|_{0,\mathcal{D}}^2 &\leq C \left[\sum_{|\beta| \leq m-1} \left(\|\psi^{|\beta|} D^\beta p\|_{2,\mathcal{D}}^2 + \|\psi^{|\beta|} D^\beta q\|_{1,\mathcal{D}}^2 \right) + \|\psi^m D^\alpha F\|_{0,\mathcal{D}}^2 \right] \\ &\leq C \left[\sum_{|\beta| \leq m-1} \left(\|\psi^{|\beta|} D^\beta F\|_{0,\mathcal{D}}^2 + E \|\psi^{|\beta|} D^\beta \phi\|_{1,\mathcal{D}}^2 \right) + \|\psi^m D^\alpha F\|_{0,\mathcal{D}}^2 \right], \end{aligned}$$

where the constant C only depends on the norm of ψ in $C^2(\overline{\mathcal{D}})$, the constants K, ρ_0, κ and T , and the function γ . Note that $\psi^m D^\alpha \phi \in \mathbb{H}_0^1(\mathcal{D})$. Therefore, applying Theorem 3.1 to BSPDE (3.7), we have Theorem 3.2 for $n = m$.

Step 2. Now we remove the boundedness assumption on a_x and σ_x made in Step 1.

In view of Theorem 3.1, BSPDE (1.1) and (1.3) has a unique strong solution (p, q) . Due to Assumption 3.3, we can construct (e.g., by the standard technique of mollification) two sequences a_r and σ_r with their first-order derivatives in x being bounded, which converge uniformly (w.r.t. (ω, t, x)) to a and σ , respectively, such that all a_r and σ_r satisfy assumptions 3.2, 3.3, and 3.4, with κ in assumption 3.3 being replaced with κ^2 . Then from Theorem 3.1, the following equation (for each n)

$$\begin{cases} dp_r = -\left(a_r^{ij} D_{ij} p_r + b^i D_i p_r - c p_r + \sigma_r^{ik} D_i q_r^k + \nu^k q_r^k + F\right) dt + q_r^k dW_t^k, \\ p_r|_{x \in \partial \mathcal{D}} = 0, \quad p_r|_{t=T} = \phi \end{cases}$$

has a unique strong solution $(p_r, q_r) \in \mathcal{H}_0^{2,1}(\mathcal{D})$, which satisfies estimate (3.3) with the constant C being independent of r . Then we can check that $\{(p_r, q_r)\}$ is a Cauchy sequence in the space $\mathcal{H}_0^{2,1}(\mathcal{D})$, whose limit is (p, q) . Similarly, we have that $\{(\psi^{|\alpha|} D^\alpha p_r, \psi^{|\alpha|} D^\alpha q_r)\}$ ($|\alpha| \leq n$) is also a Cauchy sequence in the space $\mathcal{H}_0^{2,1}(\mathcal{D})$, whose limit is denoted by (u_α, v_α) . Evidently, we have that $u_\alpha \in C([0, T], L^2(\mathcal{D})) \cap L^\infty([0, T], H^1(\mathcal{D}))$, and

$$\|(u_\alpha, v_\alpha)\|_{\mathcal{H}^{2,1}(\mathcal{D})}^2 + E \sup_{0 \leq t \leq T} \|u_\alpha(t, \cdot)\|_{1,\mathcal{D}}^2 \leq C \sum_{|\beta| \leq |\alpha|} \left(\|\psi^{|\beta|} D^\beta F\|_{0,\mathcal{D}}^2 + E \|\psi^{|\beta|} D^\beta \phi\|_{1,\mathcal{D}}^2 \right). \quad (3.9)$$

On the other hand, for every $\eta \in C_0^\infty(\mathcal{D})$ and a.e. (ω, t) , we have ($|\alpha| \leq n$)

$$\langle \psi^{|\alpha|} D^\alpha p_r, \eta \rangle = (-1)^{|\alpha|} \langle p_r, D^\alpha (\psi^{|\alpha|} \eta) \rangle \rightarrow (-1)^{|\alpha|} \langle p, D^\alpha (\psi^{|\alpha|} \eta) \rangle = \langle \psi^{|\alpha|} D^\alpha p, \eta \rangle,$$

where we denote $\langle u, v \rangle = \int_{\mathcal{D}} u(x) v(x) dx$. Thus $\psi^{|\alpha|} D^\alpha p = u_\alpha$. Similarly, we have $\psi^{|\alpha|} D^\alpha q = v_\alpha$. Estimate (3.5) is derived from inequality (3.9).

It remains to remove the additional assumption made at the beginning that $\psi \in C_b^{m+2}(\overline{\mathcal{D}})$. Note that the constant C in our estimate only depends on $|\psi|_{C^2}$ (and other parameters), which allows us to approximate ψ in $C_b^2(\overline{\mathcal{D}})$ by a sequence of nonnegative $C_b^{m+2}(\overline{\mathcal{D}})$ -functions vanishing on the boundary. Then in view of Lebesgue's dominated convergence theorem, the proof is complete. \square

By choosing a proper weighting function ψ , we obtain the following interior spacial regularity for the strong solution of BSPDE (1.1) and (1.3).

Corollary 3.4. *Let Assumptions 3.1 and 3.2 be satisfied. In addition, suppose that*

$$\sum_{|\alpha| \leq n} (|D^\alpha a^{ij}| + |D^\alpha b^i| + |D^\alpha c| + |D^\alpha \sigma^{ik}| + |D^\alpha \nu^k|) \leq K, \quad (3.10)$$

$$F \in \mathbb{H}^n(\mathcal{D}), \quad \phi \in L^2(\Omega, \mathcal{F}_T, H_0^1(\mathcal{D}) \cap H^{n+1}(\mathcal{D})).$$

Here the integer $n \geq 1$. Then BSPDE (1.1) and (1.3) has a unique strong solution $(p, q) \in \mathcal{H}_0^{2,1}(\mathcal{D})$ such that

- (i) the functions p and q satisfy (3.6) and (3.3);
- (ii) $p \in \mathbb{H}_{loc}^{n+2}(\mathcal{D})$, $q \in \mathbb{H}_{loc}^{n+1}(\mathcal{D}; \mathbb{R}^{d_1})$, $p \in C([0, T], H_{loc}^n(\mathcal{D}))(a.s.)$, i.e., for any domain $\mathcal{D}' \subset \subset \mathcal{D}$, we have

$$\begin{aligned} & \| \| p \| \|_{n+2, \mathcal{D}'}^2 + \| \| q \| \|_{n+1, \mathcal{D}'}^2 + E \sup_{0 \leq t \leq T} \| p(t, \cdot) \|_{n+1, \mathcal{D}'}^2 \\ & \leq C \sum_{m=-1}^n (\rho \wedge 1)^{-2(n-m)} (\| \| F \| \|_{m, \mathcal{D}}^2 + E \| \phi \|_{m+1, \mathcal{D}}^2), \end{aligned} \quad (3.11)$$

with $\rho = \text{dist}(\mathcal{D}', \partial \mathcal{D})$ and with the constant $C = C(K, \rho_0, \kappa, T)$;

- (iii) moreover, if $n - d/2 > 2$, then

$$\begin{aligned} p & \in L^2(\Omega \times (0, T), \mathcal{P}, C^2(\mathcal{D}')) \cap L^2(\Omega, C([0, T] \times \mathcal{D}')), \\ q & \in L^2(\Omega \times (0, T), \mathcal{P}, C^1(\mathcal{D}'; \mathbb{R}^{d_1})), \end{aligned}$$

for any domain $\mathcal{D}' \subset \subset \mathcal{D}$.

Proof. In view of Theorem 3.1, it remains to prove the assertions (ii) and (iii).

Without loss of generality, we suppose $\rho \leq 1$. Define

$$\mathcal{K} = \{x \in \mathcal{D} : \text{dist}(x, \partial \mathcal{D}) \geq \rho/2\}.$$

It is clear that $\mathcal{D}' \subset \mathcal{K}$. Take a nonnegative function $\zeta \in C_0^\infty(\mathbb{R}^d)$ such that

$$\text{supp} \zeta \subset B_{\frac{\rho}{2}}(0), \quad \int_{\mathbb{R}^d} \zeta = 1, \quad |D\zeta| \leq C\rho^{-1}, \quad |D^2\zeta| \leq C\rho^{-2}.$$

Define $\psi = (\rho/2)\zeta * 1_{\mathcal{K}}$. It is not hard to show that ψ is a well defined weight function for Theorem 3.2, and moreover

$$|\psi| \leq \frac{\rho}{2}, \quad |D\psi| \leq C, \quad |D^2\psi| \leq C\rho^{-1}, \quad \psi|_{\mathcal{D}} = \frac{\rho}{2}. \quad (3.12)$$

In view of (3.8) and keeping in mind (3.12), we show that for $n = 1$,

$$\begin{aligned} & \| \| \psi Dp \| \|_{2, \mathcal{D}}^2 + \| \| \psi Dq \| \|_{1, \mathcal{D}}^2 + E \sup_{0 \leq t \leq T} \| \psi Dp(t, \cdot) \|_{1, \mathcal{D}}^2 \\ & \leq C(\rho^{-2} \| \| D_x p \| \|_{0, \mathcal{D}}^2 + \| \| p \| \|_{2, \mathcal{D}}^2 + \| \| q \| \|_{1, \mathcal{D}}^2 + \| \| \psi DF \| \|_{0, \mathcal{D}}^2 + \| \| \psi D\phi \| \|_{1, \mathcal{D}}^2), \end{aligned}$$

and for $2 \leq m \leq n$ and multi-index α s.t. $|\alpha| = m$,

$$\begin{aligned} & \| \| \psi^m D^\alpha p \| \|_{2, \mathcal{D}}^2 + \| \| \psi^m D^\alpha q \| \|_{1, \mathcal{D}}^2 + E \sup_{0 \leq t \leq T} \| \psi^m D^\alpha p(t, \cdot) \|_{1, \mathcal{D}}^2 \\ & \leq C \left[\sum_{|\beta| \leq m-1} (\| \| \psi^{|\beta|} D^\beta p \| \|_{2, \mathcal{D}}^2 + \| \| \psi^{|\beta|} D^\beta q \| \|_{1, \mathcal{D}}^2) \right. \\ & \quad \left. + \| \| \psi^m D^\alpha F \| \|_{0, \mathcal{D}}^2 + E \| \psi^m D^\alpha \phi \|_{1, \mathcal{D}}^2 \right]. \end{aligned}$$

By induction, we have (here $|\alpha| = n$)

$$\begin{aligned} & \| \psi^n D^\alpha p \|_{2,\mathcal{D}}^2 + \| \psi^n D^\alpha q \|_{1,\mathcal{D}}^2 + E \sup_{0 \leq t \leq T} \| \psi^n D^\alpha p(t, \cdot) \|_{1,\mathcal{D}}^2 \\ & \leq C \rho^{-2} \| D_x p \|_{0,\mathcal{D}}^2 + C \sum_{|\beta| \leq n} (\| \psi^{|\beta|} D^\beta F \|_{0,\mathcal{D}}^2 + \| \psi^{|\beta|} D^\beta \phi \|_{1,\mathcal{D}}^2). \end{aligned}$$

By multiplying ρ^{-2n} on both sides, we can easily get (3.11). The assertion (iii) follows from the Sobolev embedding theorem. The proof is complete. \square

Remark 3.2. In the case of $n - d/2 > 2$, the function pair (p, q) satisfies equation (2.2) for all $(t, x) \in [0, T] \times \mathcal{D}$ and $\omega \in \Omega'$ s.t. $\mathbb{P}(\Omega') = 1$, which is a classical solution of equation (1.1) with the terminal-boundary condition (1.3) (see e.g. [15] for details).

4 Proof of Theorem 3.1

The proof is rather long, and it is divided into two subsections. We first consider the simpler domain of a half space, and then go to the general C^2 domain.

4.1 The case of the half space

In this subsection, we consider BSPDE (1.1) and (1.3) living in a half space.

Recall $\mathbb{R}_+^d = \{x \in \mathbb{R}^d : x^1 > 0\}$. Denote $y = (x^2, \dots, x^d)$.

Definition 4.1. We say a function f defined on \mathbb{R}^d has the *property of reflection invariance*, if $f(x^1, y) = -f(-x^1, y)$ for almost every $(x^1, y) \in \mathbb{R}^d$.

For a function u defined on \mathbb{R}_+^d , define \tilde{u} and \bar{u} as follows:

$$\tilde{u} = \begin{cases} u, & \text{on } \{x^1 > 0\}, \\ 0, & \text{on } \{x^1 \leq 0\}; \end{cases} \quad \bar{u}(x^1, y) = \begin{cases} u(x^1, y), & \text{if } x^1 > 0, \\ 0, & \text{if } x^1 = 0, \\ -u(-x^1, y), & \text{if } x^1 < 0. \end{cases} \quad (4.1)$$

It is clear that \bar{u} has the *property of reflection invariance*.

Lemma 4.1. (a) *Let m be a positive number. Then a function $u \in H_0^m(\mathbb{R}_+^d)$ if and only if $\tilde{u} \in H^m(\mathbb{R}^d)$.*

(b) *The function $u \in H_0^1(\mathbb{R}_+^d)$ if and only if $\bar{u} \in H^1(\mathbb{R}^d)$.*

Proof. The proof of assertion (a) can be found in, e.g., Chen [3, Page 48]. The necessity of assertion (b) follows from assertion (a). It remains to prove the sufficiency of (b). Indeed, we can find $\varphi_n \in C_0^\infty(\mathbb{R}^d)$ such that $\varphi_n \rightarrow \bar{u}$ in $H^1(\mathbb{R}^d)$ as $n \rightarrow \infty$. Denote $\bar{\varphi}_n(x^1, y) = -\varphi_n(-x^1, y)$. Note that $\bar{u}(x^1, y) = -\bar{u}(-x^1, y)$. Thus we have $\bar{\varphi}_n \rightarrow \bar{u}$ in $H^1(\mathbb{R}^d)$. Define $\psi_n = (\varphi_n + \bar{\varphi}_n)/2$. Then $\psi_n \rightarrow \bar{u}$ in $H^1(\mathbb{R}^d)$. Since $\psi_n(0, y) = 0$, the restriction of ψ_n on \mathbb{R}_+^d belongs to $H_0^1(\mathbb{R}_+^d)$. Thus $u \in H_0^1(\mathbb{R}_+^d)$. The proof is complete. \square

The following existence and uniqueness result concerning the Cauchy problem of BSPDEs is taken from Du and Meng [5, Prop. 4.1], which can also be proved by means of the duality method of Zhou [24] and Lemma 2.1.

Lemma 4.2. Consider the following BSPDE (on \mathbb{R}^d)

$$\begin{cases} dp = -[a^{ij}(t)D_{ij}p + \sigma^{ik}(t)D_i q^k + F]dt + q^k dW_t^k, \\ p(T, x) = \phi(x), \quad x \in \mathbb{R}^d, \end{cases} \quad (4.2)$$

where a and σ are two predictable processes taken values in S^n and $\mathbb{R}^{d \times d_1}$, respectively, such that $\kappa I + \sigma \sigma^* \leq 2a \leq \kappa^{-1}I$, $\forall (\omega, t)$. Suppose $F \in \mathbb{H}^0(\mathbb{R}^d)$, $\phi \in L^2(\Omega, \mathcal{F}_T, H^1(\mathbb{R}^d))$. Then BSPDE (4.2) has a unique strong solution (p, q) in $\mathbb{H}^2(\mathbb{R}^d) \times \mathbb{H}^1(\mathbb{R}^d; \mathbb{R}^{d_1})$ such that $p \in C([0, T], H^1(\mathbb{R}^d))$ (a.s.), and moreover,

$$\| \| p \| \|_{2, \mathbb{R}^d}^2 + \| \| q \| \|_{1, \mathbb{R}^d}^2 + E \sup_{0 \leq t \leq T} \| p(t, \cdot) \|_{1, \mathbb{R}^d}^2 \leq C(\kappa, T) (\| \| F \| \|_{0, \mathbb{R}^d}^2 + E \| \phi \|_{1, \mathbb{R}^d}^2). \quad (4.3)$$

Now we have the following

Lemma 4.3. Let Assumptions 3.2 be satisfied with $\mathcal{D} = \mathbb{R}_+^d$. Assume that a and σ are invariant in the space variable x . Suppose that $F \in \mathbb{H}^0(\mathbb{R}_+^d)$ and $\phi \in L^2(\Omega, \mathcal{F}_T, H_0^1(\mathbb{R}_+^d))$. Then the following BSPDE

$$\begin{cases} dp = -[a^{ij}(t)D_{ij}p + \sigma^{ik}(t)D_i q^k + F]dt + q^k dW_t^k, \\ p(t, x) = 0, \quad x \in \partial \mathbb{R}_+^d, \\ p(T, x) = \phi(x), \quad x \in \mathbb{R}_+^d \end{cases} \quad (4.4)$$

has a unique strong solution $(p, q) \in \mathcal{H}_0^{2,1}(\mathbb{R}_+^d)$ such that $p \in C([0, T], H^1(\mathbb{R}_+^d))$ (a.s.), and

$$\| (p, q) \|_{\mathcal{H}^{2,1}(\mathbb{R}_+^d)}^2 + E \sup_{0 \leq t \leq T} \| p(t, \cdot) \|_{1, \mathbb{R}_+^d}^2 \leq C (\| \| F \| \|_{0, \mathbb{R}_+^d}^2 + E \| \phi \|_{1, \mathbb{R}_+^d}^2), \quad (4.5)$$

where the constant C depends only on κ and T .

Proof. Recalling the definition (4.1), we have $\overline{F} \in \mathbb{H}^0(\mathbb{R}^d)$ and $\overline{\phi} \in L^2(\Omega, \mathcal{F}_T, H^1(\mathbb{R}^d))$. From Lemma 4.2, the BSPDE

$$\begin{cases} dP = -[a^{ij}(t)D_{ij}P + \sigma^{ik}(t)D_i Q^k + \overline{F}]dt + Q^k dW_t^k, \\ P(T, x) = \overline{\phi}(x), \quad x \in \mathbb{R}^d \end{cases} \quad (4.6)$$

has a unique strong solution (P, Q) in the space $\mathbb{H}^2(\mathbb{R}^d) \times \mathbb{H}^1(\mathbb{R}^d; \mathbb{R}^{d_1})$ such that $P \in C([0, T], H^1(\mathbb{R}^d))$ (a.s.), with the estimate

$$\| \| P \| \|_{2, \mathbb{R}^d}^2 + \| \| Q \| \|_{1, \mathbb{R}^d}^2 + E \sup_{0 \leq t \leq T} \| P(t, \cdot) \|_{1, \mathbb{R}^d}^2 \leq C(\kappa, T) (\| \| \overline{F} \| \|_{0, \mathbb{R}^d}^2 + E \| \overline{\phi} \|_{1, \mathbb{R}^d}^2). \quad (4.7)$$

By symmetry and the uniqueness of the solution (of equation (4.6)), we know that P and Q have the property of reflection invariance, for a.e. (ω, t) . Denote by p and q the restrictions of P and Q on \mathbb{R}_+^d , respectively. From Lemma 4.1 (b), we know that $p \in \mathbb{H}_0^1 \cap \mathbb{H}^2(\mathbb{R}_+^d)$ and $q \in \mathbb{H}_0^1(\mathbb{R}_+^d; \mathbb{R}^{d_1})$. Moreover, $p \in C([0, T], H^1(\mathbb{R}_+^d))$ (a.s.). It is evident that the pair (p, q) is a strong solution of equation (4.4). Since every strong solution of equation (4.4) is also a weak solution, from the uniqueness of the weak solution, we know that (p, q) is the unique strong solution of (4.4). Estimate (4.5) follows from inequality (4.7). The proof is complete. \square

Now we prove the following perturbation result, which will be used in the proof of Theorem 3.1.

Proposition 4.4. *Consider the following BSPDE*

$$\begin{cases} dp = -[a^{ij}D_{ij}p + \sigma^{ik}D_iq^k + F]dt + q^k dW_t^k, \\ p(t, x) = 0, \quad x \in \partial\mathbb{R}_+^d, \\ p(T, x) = \phi(x), \quad x \in \mathbb{R}_+^d. \end{cases} \quad (4.8)$$

Assume that for a constant $\delta > 0$ and for any (ω, t, x) we have

$$|a(t, x) - a_0(t)| \leq \delta, \quad |\sigma(t, x) - \sigma_0(t)| \leq \delta, \quad (4.9)$$

where $\{a_0(t) : 0 \leq t \leq T\}$ and $\{\sigma_0(t) : 0 \leq t \leq T\}$ are predictable processes satisfying Assumptions 3.2. Suppose that $F \in \mathbb{H}^0(\mathbb{R}_+^d)$ and $\phi \in L^2(\Omega, \mathcal{F}_T, H_0^1(\mathbb{R}_+^d))$.

Under the above assumptions, we assert that there exists a constant $\delta(\kappa, T) > 0$ such that if $\delta \leq \delta(\kappa, T)$ then BSPDE (4.8) has a unique strong solution $(p, q) \in \mathcal{H}_0^{2,1}(\mathbb{R}_+^d)$ such that $p \in C([0, T], H^1(\mathbb{R}_+^d))$ (a.s.), and

$$\|(p, q)\|_{\mathcal{H}^{2,1}(\mathbb{R}_+^d)}^2 + E \sup_{0 \leq t \leq T} \|p(t, \cdot)\|_{1, \mathbb{R}_+^d}^2 \leq C(\|F\|_{0, \mathbb{R}_+^d}^2 + E\|\phi\|_{1, \mathbb{R}_+^d}^2), \quad (4.10)$$

where the constant C depends on κ and T .

Proof. In view of Lemma 4.3, we know that for any $(u, v) \in \mathcal{H}^{2,1}(\mathbb{R}_+^d)$, the BSPDE

$$\begin{cases} dp = -[a_0^{ij}D_{ij}p + \sigma_0^iD_iq + (a^{ij} - a_0^{ij})D_{ij}u \\ \quad + (\sigma^i - \sigma_0^i)D_iv + F]dt + qdW_t, \\ p(t, x) = 0, \quad x \in \partial\mathbb{R}_+^d, \\ p(T, x) = \phi(x), \quad x \in \mathbb{R}_+^d \end{cases} \quad (4.11)$$

has a unique strong solution $(p, q) \in \mathcal{H}_0^{2,1}(\mathbb{R}_+^d)$ such that $p \in C([0, T], H^1(\mathbb{R}_+^d))$ (a.s.). We define the operator $A : \mathcal{H}^{2,1}(\mathbb{R}_+^d) \rightarrow \mathcal{H}^{2,1}(\mathbb{R}_+^d)$ as follows:

$$A(u, v) = (p, q).$$

Then from estimate (4.5), we obtain that for $(u_i, v_i) \in \mathcal{H}^{2,1}(\mathbb{R}_+^d)$, $i = 1, 2$,

$$\|A(u_1 - u_2, v_1 - v_2)\|_{\mathcal{H}^{2,1}(\mathbb{R}_+^d)}^2 \leq C\delta\|(u_1 - u_2, v_1 - v_2)\|_{\mathcal{H}^{2,1}(\mathbb{R}_+^d)}^2.$$

Taking $\delta = (2C)^{-1} = (2C(\kappa, T))^{-1}$, we have that the operator A is a contraction in $\mathcal{H}^{2,1}(\mathbb{R}_+^d)$, and thus there exists a unique element $(p, q) \in \mathcal{H}^{2,1}(\mathbb{R}_+^d)$ such that $A(p, q) = (p, q)$. Furthermore, we have $(p, q) \in \mathcal{H}_0^{2,1}(\mathbb{R}_+^d)$ and $p \in C([0, T], H^1(\mathbb{R}_+^d))$ (a.s.). It is clear that (p, q) is the unique strong solution of BSPDE (4.8).

To establish estimate (4.10), in view of Lemma 4.3, applying estimate (4.5) to equation (4.11), we have

$$\|(p, q)\|_{\mathcal{H}^{2,1}(\mathbb{R}_+^d)}^2 + E \sup_{0 \leq t \leq T} \|p(t, \cdot)\|_{1, \mathbb{R}_+^d}^2 \leq C\delta\|(p, q)\|_{\mathcal{H}^{2,1}(\mathbb{R}_+^d)}^2 + C(\|F\|_{0, \mathbb{R}_+^d}^2 + E\|\phi\|_{1, \mathbb{R}_+^d}^2).$$

Taking $\delta = (2C)^{-1}$, we obtain (4.10). The proof is complete. \square

4.2 The case of the general C^2 domain

In this subsection, we shall complete the proof of Theorem 3.1. To simplify the notation, we define

$$\mathfrak{H}^{2,1}(\mathcal{D}) = \left\{ (u, v) \in \mathcal{H}_0^{2,1}(\mathcal{D}) : u \in L^2(\Omega, L^\infty([0, T], H^1(\mathcal{D}))), \right. \\ \left. u \in C([0, T], L^2(\mathcal{D})) \text{ (a.s.)} \right\}$$

being equipped with the norm

$$\|(u, v)\|_{\mathfrak{H}^{2,1}(\mathcal{D})} = \left(\|(u, v)\|_{\mathcal{H}^{2,1}(\mathcal{D})}^2 + E \sup_{0 \leq t \leq T} \|u(t, \cdot)\|_{1, \mathcal{D}}^2 \right)^{1/2}. \quad (4.12)$$

It is clear that $\mathfrak{H}^{2,1}(\mathcal{D})$ is a Banach space.

First we have the following fact.

Lemma 4.5. *Let Φ be the map defined in Assumption 3.1 and Ψ be the inverse of Φ . Suppose $u \in H_0^1(\mathbb{R}_+^d)$ s.t. $\text{supp}(u) \subset B_+ \cup \partial\mathbb{R}_+^d$. Then $u \circ \Psi \in H_0^1(\mathcal{D})$.*

Proof. The proof is straightforward. Take $u_n \in C_0^\infty(B_+)$ such that $u_n \rightarrow u$ strongly in $H^1(\mathbb{R}_+^d)$ as $n \rightarrow \infty$. From the properties of Φ , it is easy to show that $u_n \circ \Psi \in C^2(\mathcal{D})$ and $\text{supp}(u_n \circ \Psi) \subset U \cup \mathcal{D}$ and then $u_n \circ \Psi \in H_0^1(\mathcal{D})$, where U is taken from Assumption 3.1. Now we have

$$\|u_n \circ \Psi - u \circ \Psi\|_{0, \mathcal{D}}^2 \leq |\det(D\Phi)|_{L^\infty} \|u_n - u\|_{0, B_+}^2 \rightarrow 0, \\ \|D(u_n \circ \Psi - u \circ \Psi)\|_{0, \mathcal{D}}^2 \leq |\det(D\Phi)|_{L^\infty} |D\Psi|_{L^\infty}^2 \|D(u_n - u)\|_{0, B_+}^2 \rightarrow 0,$$

as $n \rightarrow \infty$, which implies $u \circ \Psi \in H_0^1(\mathcal{D})$. \square

The following is Theorem 3.1 under an additional assumption on the coefficients a and σ .

Proposition 4.6. *Let the conditions of Theorem 3.1 be satisfied. In addition, assume that a_x and σ_x are bounded. Then BSPDE (1.1) and (1.3) has a unique strong solution $(p, q) \in \mathfrak{H}^{2,1}(\mathcal{D})$ such that*

$$\| \| p \| \|_{2, \mathcal{D}}^2 + \| \| q \| \|_{1, \mathcal{D}}^2 + E \sup_{0 \leq t \leq T} \| p(t, \cdot) \|_{1, \mathcal{D}}^2 \leq C (\| \| F \| \|_{0, \mathcal{D}}^2 + E \| \phi \|_{1, \mathcal{D}}^2), \quad (4.13)$$

where the constant C only depends K, ρ_0, κ, T , and the function γ .

Proof. Since a_x and σ_x are bounded, equation (1.1) can be written into the divergence form

$$dp = -[(a^{ij} p_{x^j} + \sigma^{ik} q^k)_{x^i} + (b^i - a_{x^j}^{ij}) p_{x^i} - cp + (\nu^k - \sigma_{x^i}^{ik}) q^k + F] dt + q^k dW_t^k.$$

From Lemma 2.3, BSPDE (1.1) and (1.3) has a unique weak solution $(p, q) \in \mathbb{H}_0^1(\mathcal{D}) \times \mathbb{H}^0(\mathcal{D}; \mathbb{R}^{d_1})$.

Now take a sufficiently small $\rho \in (0, \rho_0)$ to satisfy the following two conditions.

- (1) $\gamma(8\rho) \leq \delta$ with the constant $\delta = \delta(\kappa, T)$ being given by Proposition 4.4. In view of Assumption 3.3, for any (ω, t) and $x, y \in \mathcal{D}$, we have

$$|a(t, x) - a(t, y)| \leq \delta, \quad |\sigma(t, x) - \sigma(t, y)| \leq \delta \quad (4.14)$$

if $|x - y| \leq 8\rho$.

- (2) If x, y belong to the same domain U in Assumption 3.1, then for any (ω, t) ,

$$\begin{aligned} |D\Psi(x)a(t, x)(D\Psi(x))^* - D\Psi(y)a(t, y)(D\Psi(y))^*| &\leq \delta_1, \\ |D\Psi(x)\sigma(t, x) - D\Psi(y)\sigma(t, y)| &\leq \delta_1 \end{aligned} \quad (4.15)$$

if $|x - y| \leq 8\rho$, where the constant $\delta_1 = \delta(\kappa^2, T)$ and Ψ is the inverse of Φ .

Then take a nonnegative function $\zeta \in C_0^\infty(\mathbb{R}^d)$ such that $\text{supp}(\zeta) \subset B_{2\rho}(0)$, $\zeta(x) = 1$ for $|x| \leq \rho$. For any $z \in \mathbb{R}^d$, define for $x \in \mathbb{R}^d$,

$$\zeta^z(x) = \zeta(x - z), \quad p^z(t, x) = p(t, x)\zeta^z(x), \quad q^z(t, x) = q(t, x)\zeta^z(x). \quad (4.16)$$

Then (p^z, q^z) satisfies the following equation (in the sense of Definition 2.1 (ii))

$$\begin{aligned} dp^z = & - \left[a^{ij} p_{x^i x^j}^z + \sigma^{ik} q_{x^i}^{z,k} + \zeta^z F + (b^i \zeta^z - 2a^{ij} \zeta_{x^j}^z) p_{x^i} \right. \\ & \left. - (a^{ij} \zeta_{x^i x^j}^z + c) \zeta^z p + (\nu^k - \sigma^{ik} \zeta_{x^i}^z) \zeta^z q^k \right] dt + q^{z,k} dW_t^k. \end{aligned} \quad (4.17)$$

In addition, define

$$\eta^z(x) = \zeta\left(\frac{x - z}{2}\right).$$

Now consider the following two cases.

Case 1. $\text{dist}(z, \partial\mathcal{D}) \leq 2\rho_0$. Then $\mathcal{D} \cap \text{supp}(p^z) \subset U \cap \mathcal{D}$, where U is given in Assumption 3.1. Set $x = \Phi(y)$. Define

$$u^z(t, y) = p^z(t, x), \quad v^z(t, y) = q^z(t, x), \quad (t, y) \in (0, T) \times \mathbb{R}_{y,+}^d.$$

Obviously $(u^z, v^z) \in \mathbb{H}_0^1(\mathbb{R}_{y,+}^d) \times \mathbb{H}^0(\mathbb{R}_{y,+}^d; \mathbb{R}^{d_1})$. Direct calculus shows that

$$\begin{aligned} p_{x^r x^s}^z(t, x) &= \Psi_{x^r}^i(x) \Psi_{x^s}^j(x) u_{y^i y^j}^z(t, y) + (\zeta^z p_{x^i} + \zeta_{x^i}^z p)(t, x) \Psi_{x^r x^s}^i(x) \Phi_{y^i}^r(y) \\ q_{x^r}^{z,k}(t, x) &= \Psi_{x^r}^i(x) v_{y^i}^{z,k}(t, y). \end{aligned}$$

Substituting the above relations into equation (4.17), it is not hard to check that the functions u^z, v^z satisfy the BSPDE (in the sense of Definition 2.1 (ii))

$$\begin{cases} du^z = -(\tilde{a}^{ij} u_{y^i y^j}^z + \tilde{\sigma}^{ik} v_{y^i}^{z,k} \tilde{F}) dt + v^{z,k} dW_t^k, \\ u^z|_{\mathbb{R}_{y,+}^d} = 0, \quad u^z|_{t=T} = \zeta^z \phi, \end{cases} \quad (4.18)$$

where (observe that $u^z = 0, v^z = 0$ whenever $\eta^z \neq 1$)

$$\begin{aligned} x &= \Phi(y), \quad x_0 = \Phi(0), \quad L^0 = a^{rs} \partial_{x^r x^s}^2, \\ \tilde{a}^{ij}(t, y) &= a^{rs}(t, x) \Psi_{x^r}^i \Psi_{x^s}^j(x) \eta^z(x) + a^{rs}(t, x_0) \Psi_{x^r}^i \Psi_{x^s}^j(x_0) (1 - \eta^z(x)), \\ \tilde{\sigma}^{ik}(t, y) &= \sigma^{rk}(t, x) \Psi_{x^r}^i(x) \eta^z(x) + \sigma^{rk}(t, x_0) \Psi_{x^r}^i(x_0) (1 - \eta^z(x)), \\ \tilde{F}(t, y) &= (\zeta^z F)(t, x) + p_{x^r}(t, x) \Theta_1^r(t, y) + p(t, x) \Theta_2(t, y) + q^k(t, x) \Theta_3^k(t, y), \\ \Theta_1^r(t, y) &= (\zeta^z L^0 \Psi^i)(t, x) \Phi_{y^i}^r(t, y) + (b^r \zeta^z - 2a^{rs} \zeta_{x^s}^z)(t, x), \\ \Theta_2(t, y) &= (\zeta_{x^r}^z L^0 \Psi^i)(t, x) \Phi_{y^i}^r(t, y) - (L^0 \zeta^z + c \zeta^z)(t, x), \\ \Theta_3^k(t, y) &= (\nu^k \zeta^z - \sigma^{rk} \zeta_{x^r}^z)(t, x). \end{aligned}$$

In order to apply Proposition 4.4 to BSPDE (4.18), we take

$$a_0(t) = \tilde{a}(t, 0), \quad \sigma_0(t) = \tilde{\sigma}(t, 0). \quad (4.19)$$

Note that $\text{supp}(\eta^z) \subset B_{4\rho}(z)$. Then it follows from (4.15) that for any $y \in \mathbb{R}_y^d$ and $x = \Phi(y)$, we have (recall $x_0 = \Phi(0)$)

$$\begin{aligned} |\tilde{a}(t, y) - \tilde{a}(t, 0)| &= |a^{rs}(t, x)\Psi_{x^r}^i \Psi_{x^s}^j(x) - a^{rs}(t, x_0)\Psi_{x^r}^i \Psi_{x^s}^j(x_0)| \cdot |\eta^z(x)| \\ &\leq |a^{rs}(t, x)\Psi_{x^r}^i \Psi_{x^s}^j(x) - a^{rs}(t, x_0)\Psi_{x^r}^i \Psi_{x^s}^j(x_0)| \cdot 1_{B_{4\rho}(z)} \\ &\leq \delta_1 = \delta(\kappa^2, T). \end{aligned}$$

Therefore, from Proposition 4.4, BSPDE (4.18) has a unique strong solution (u, v) such that

$$u \in \mathbb{H}_0^1 \cap \mathbb{H}^2(\mathbb{R}_{y,+}^d), \quad v \in \mathbb{H}_0^1(\mathbb{R}_{y,+}^d; \mathbb{R}^{d_1}).$$

It is clear that (u, v) is also a weak solution to BSPDE (4.18). From the uniqueness of the weak solution, we have $u = u^z$ and $v = v^z$. Hence we deduce that

$$u^z \in \mathbb{H}_0^1 \cap \mathbb{H}^2(\mathbb{R}_{y,+}^d), \quad v^z \in \mathbb{H}_0^1(\mathbb{R}_{y,+}^d; \mathbb{R}^{d_1}). \quad (4.20)$$

Denote $D(z, r) = B_r(z) \cap \mathcal{D}$. Then (4.20) implies that restricted on the domain $D(z, \rho)$, the solution

$$(p, q) \in \mathbb{H}^2(D(z, \rho)) \times \mathbb{H}^1(D(z, \rho); \mathbb{R}^{d_1}) \quad (4.21)$$

for any z s.t. $\text{dist}(z, \partial\mathcal{D}) \leq 2\rho_0$.

Now applying estimate (4.10) to BSPDE (4.18), we obtain that

$$\begin{aligned} &\| \| u^z \| \|_{2, \mathbb{R}_{y,+}^d}^2 + \| \| v^z \| \|_{1, \mathbb{R}_{y,+}^d}^2 + E \sup_{0 \leq t \leq T} \| u^z(t, \cdot) \|_{1, \mathbb{R}_{y,+}^d}^2 \\ &\leq C(\kappa, T) (\| \tilde{F} \|_{0, \mathbb{R}_{y,+}^d}^2 + E \| (\zeta^z \phi) \circ \Phi \|_{1, \mathbb{R}_{y,+}^d}^2). \end{aligned}$$

On the other hand, it is evident that (recall $\text{supp}(\zeta^z) \subset B_{2\rho}(z)$)

$$\begin{aligned} &\| \| p \| \|_{2, D(z, \rho)}^2 + \| \| q \| \|_{1, D(z, \rho)}^2 + E \sup_{0 \leq t \leq T} \| p(t, \cdot) \|_{1, D(z, \rho)}^2 \\ &\leq \| \| p^z \| \|_{2, \mathcal{D}}^2 + \| \| q^z \| \|_{1, \mathcal{D}}^2 + E \sup_{t \leq T} \| p^z(t, \cdot) \|_{1, \mathcal{D}}^2 \\ &\leq C (\| \| u^z \| \|_{2, \mathbb{R}_{y,+}^d}^2 + \| \| v^z \| \|_{1, \mathbb{R}_{y,+}^d}^2 + E \sup_{0 \leq t \leq T} \| u^z(t, \cdot) \|_{1, \mathbb{R}_{y,+}^d}^2), \\ &\| \tilde{F} \|_{0, \mathbb{R}_{y,+}^d}^2 \leq C (\| F \|_{0, D(z, 2\rho)}^2 + \| \| p \| \|_{1, D(z, 2\rho)}^2 + \| \| q \| \|_{0, D(z, 2\rho)}^2), \\ &E \| (\zeta^z \phi) \circ \Phi \|_{1, \mathbb{R}_{y,+}^d}^2 \leq CE \| \zeta^z \phi \|_{1, \mathcal{D}}^2 \leq CE \| \phi \|_{1, D(z, 2\rho)}^2, \end{aligned}$$

where the constant C depends only on K and κ . Therefore, we obtain

$$\begin{aligned} &\| \| p \| \|_{2, D(z, \rho)}^2 + \| \| q \| \|_{1, D(z, \rho)}^2 + E \sup_{0 \leq t \leq T} \| p(t, \cdot) \|_{1, D(z, \rho)}^2 \\ &\leq C (\| F \|_{0, D(z, 2\rho)}^2 + E \| \phi \|_{1, D(z, 2\rho)}^2 + \| \| p \| \|_{1, D(z, 2\rho)}^2 + \| \| q \| \|_{0, D(z, 2\rho)}^2). \end{aligned} \quad (4.22)$$

Case 2. $\text{dist}(z, \partial\mathcal{D}) \geq 2\rho_0$. This case can easily be reduced to the first one. Indeed, we can replace the domain \mathcal{D} by any half space with the boundary lying at a distance

$2\rho_0$ from z . In this situation it is not necessary to flatten the boundary and to change coordinates. Then as above we deduce property (4.21) for any $z \in \mathcal{D}$ and obtain an estimate similar to (4.22).

Integrating both sides of inequality (4.22) over $z \in \mathbb{R}^d$, we obtain that

$$\begin{aligned} & \| \| p \|_{2,\mathcal{D}}^2 + \| \| q \|_{1,\mathcal{D}}^2 + E \sup_{0 \leq t \leq T} \| p(t, \cdot) \|_{1,\mathcal{D}}^2 \\ & \leq C (\| \| F \|_{0,\mathcal{D}}^2 + E \| \phi \|_{1,\mathcal{D}}^2 + \| \| p \|_{1,\mathcal{D}}^2 + \| \| q \|_{0,\mathcal{D}}^2), \end{aligned} \quad (4.23)$$

where the constant C depends on K, ρ_0, κ, T , and the function γ . Since $(p, q) \in \mathbb{H}_0^1(\mathcal{D}) \times \mathbb{H}^0(\mathcal{D}; \mathbb{R}^{d_1})$, the right-hand side is finite. Recalling that (4.21) holds true for any $z \in \mathcal{D}$, the above estimate implies that the unique weak solution of BSPDE (1.1) and (1.3) found by Lemma 2.3 belongs to the space $\mathcal{H}^{2,1}(\mathcal{D})$, and moreover, $p \in C([0, T], L^2(\mathcal{D})) \cap L^\infty([0, T], H^1(\mathcal{D}))$ (a.s.). From Proposition 2.2, we know that (p, q) is the unique strong solution of BSPDE (1.1) and (1.3).

Replace the initial time zero by any $s \in [0, T]$. Proceeding identically as before, we can obtain the following estimate similar to (4.23)

$$\begin{aligned} & E \int_s^T \| p(t, \cdot) \|_{2,\mathcal{D}}^2 dt + E \int_s^T \| q(t, \cdot) \|_{1,\mathcal{D}}^2 dt + E \sup_{s \leq t \leq T} \| p(t, \cdot) \|_{1,\mathcal{D}}^2 \\ & \leq C \left(\| \| F \|_{0,\mathcal{D}}^2 + E \| \phi \|_{1,\mathcal{D}}^2 + E \int_s^T \| p(t, \cdot) \|_{1,\mathcal{D}}^2 dt + E \int_s^T \| q(t, \cdot) \|_{0,\mathcal{D}}^2 dt \right). \end{aligned} \quad (4.24)$$

In view of the definition of the strong solution (Definition 2.1), we know that the process $p(t, \cdot)$ is an $L^2(\mathcal{D})$ -valued semi-martingale. Then applying Itô's formula for Hilbert-valued semi-martingales (see e.g. [4, Page 105]), we have

$$\begin{aligned} \| p(s, \cdot) \|_{0,\mathcal{D}}^2 &= \| \phi \|_{0,\mathcal{D}}^2 + 2 \int_s^T \int_{\mathcal{D}} p(a^{ij} p_{x^i x^j} + b^i p_{x^i} - cp + \sigma^{ik} q_{x^i}^k + \nu^k q^k + F) dx dt \\ &\quad - \int_s^T \| q(t, \cdot) \|_{0,\mathcal{D}}^2 dt - 2 \int_s^T \int_{\mathcal{D}} p q^k dx dW_t^k. \end{aligned}$$

Taking expectations and using the Cauchy-Schwarz inequality, we have for any $\varepsilon > 0$

$$\begin{aligned} E \int_s^T \| q(t, \cdot) \|_{0,\mathcal{D}}^2 &\leq E \| \phi \|_{0,\mathcal{D}}^2 + 2E \int_s^T \int_{\mathcal{D}} p(a^{ij} p_{x^i x^j} + b^i p_{x^i} - cp + \sigma^{ik} q_{x^i}^k + \nu^k q^k + F) dx dt \\ &\leq E \| \phi \|_{0,\mathcal{D}}^2 + \varepsilon E \int_s^T (\| p(t, \cdot) \|_{2,\mathcal{D}}^2 + \| q(t, \cdot) \|_{1,\mathcal{D}}^2) dt \\ &\quad + C(\varepsilon, K) E \int_s^T \| p(t, \cdot) \|_{0,\mathcal{D}}^2 dt + \| \| F \|_{0,\mathcal{D}}^2. \end{aligned}$$

Taking ε small enough and recalling (4.24), we have

$$\begin{aligned} & E \int_s^T \| p(t, \cdot) \|_{2,\mathcal{D}}^2 dt + E \int_s^T \| q(t, \cdot) \|_{1,\mathcal{D}}^2 dt + E \sup_{s \leq t \leq T} \| p(t, \cdot) \|_{1,\mathcal{D}}^2 \\ & \leq C (\| \| F \|_{0,\mathcal{D}}^2 + E \| \phi \|_{1,\mathcal{D}}^2 + E \int_s^T \| p(t, \cdot) \|_{1,\mathcal{D}}^2 dt), \end{aligned} \quad (4.25)$$

where the constant C depends on K, ρ_0, κ, T , and the function γ . In particular, we have

$$E\|p(s, \cdot)\|_{1, \mathcal{D}}^2 \leq C \left(\|F\|_{0, \mathcal{D}}^2 + E\|\phi\|_{1, \mathcal{D}}^2 + E \int_s^T \|p(t, \cdot)\|_{1, \mathcal{D}}^2 dt \right).$$

Using the Gronwall inequality, we have

$$\|p\|_{1, \mathcal{D}}^2 = \int_0^T E\|p(s, \cdot)\|_{1, \mathcal{D}}^2 ds \leq Ce^{CT} (\|F\|_{0, \mathcal{D}}^2 + E\|\phi\|_{1, \mathcal{D}}^2).$$

The last inequality along with (4.25) yields estimate (4.13).

It remains to prove $q \in \mathbb{H}_0^1(\mathcal{D}; \mathbb{R}^{d_1})$. Since $q \in \mathbb{H}^1(\mathcal{D}; \mathbb{R}^{d_1})$, it is sufficient to check $q^z \in \mathbb{H}_0^1(\mathcal{D}; \mathbb{R}^{d_1})$ for each $z \in \partial\mathcal{D}$ (recall (4.16)). Since $v^z = q^z \circ \Phi \in \mathbb{H}_0^1(\mathbb{R}^d; \mathbb{R}^{d_1})$, by virtue of Lemma 4.5, we get $q^z \in \mathbb{H}_0^1(\mathcal{D}; \mathbb{R}^{d_1})$. The proof is complete. \square

Remark 4.1. The constant C appearing in estimate (4.13) does not depend on $|a_x|$ and $|\sigma_x|$.

Proceeding identically to the proof of Proposition 4.6, we can prove the following

Proposition 4.7. *Let the conditions of Theorem 3.1 be satisfied. In addition, assume that the function pair $(p, q) \in \mathcal{H}^{2,1}(\mathcal{D})$ is a strong solution of BSPDE (1.1) and (1.3). Then $(p, q) \in \mathfrak{H}^{2,1}(\mathcal{D})$, and there exists a constant C only depending on K, ρ_0, κ, T and the function γ such that*

$$\|(p, q)\|_{\mathfrak{H}^{2,1}(\mathcal{D})}^2 \leq C (\|F\|_{0, \mathcal{D}}^2 + E\|\phi\|_{1, \mathcal{D}}^2). \quad (4.26)$$

Now we use the standard method of continuation to prove Theorem 3.1.

Proof of Theorem 3.1. The uniqueness of the strong solution to equation (1.1) is an immediate consequence of estimate (4.26). Now we define

$$\begin{aligned} \mathcal{L}_0 &= a^{ij}(t, 0)D_{ij} + b^i(t, x)D_i - c(t, x), & \mathcal{M}_0^k &= \sigma^{ik}(t, 0)D_i + \nu^k(t, x), \\ \mathcal{L}_1 &= a^{ij}(t, x)D_{ij} + b^i(t, x)D_i - c(t, x), & \mathcal{M}_1^k &= \sigma^{ik}(t, x)D_i + \nu^k(t, x). \end{aligned}$$

For $\lambda \in [0, 1]$, define

$$\mathcal{L}_\lambda = (1 - \lambda)\mathcal{L}_0 + \lambda\mathcal{L}_1, \quad \mathcal{M}_\lambda^k = (1 - \lambda)\mathcal{M}_0^k + \lambda\mathcal{M}_1^k.$$

Consider the following equation

$$dp = -(\mathcal{L}_\lambda p + \mathcal{M}_\lambda^k q^k + F)dt + q^k dW_t^k, \quad p|_{x \in \partial\mathcal{D}} = 0, \quad p|_{t=T} = \phi. \quad (4.27)$$

It is clear that the coefficients of equation (4.27) satisfy the conditions of Theorem 3.1 with the same K, κ and γ . Hence a priori estimate (4.26) holds for equation (4.27) for each $\lambda \in [0, 1]$ with the same constant C (i.e., independent of λ).

Assume that for some $\lambda = \lambda_0 \in [0, 1]$, equation (4.27) is solvable, i.e., it has a unique strong solution (p, q) such that $(p, q) \in \mathfrak{H}^{2,1}(\mathcal{D})$ for any $F \in \mathbb{H}^0(\mathcal{D})$ and any $\phi \in L^2(\Omega, \mathcal{F}_T, H_0^1(\mathcal{D}))$. For other $\lambda \in [0, 1]$, we can rewrite (4.27) as

$$dp = -\{\mathcal{L}_{\lambda_0} p + \mathcal{M}_{\lambda_0}^k q^k + (\lambda - \lambda_0)[(\mathcal{L}_1 - \mathcal{L}_0)p + (\mathcal{M}_1^k - \mathcal{M}_0^k)q^k] + F\}dt + q^k dW_t^k.$$

Thus for any $(u, v) \in \mathcal{H}^{2,1}(\mathcal{D})$, the equation

$$dp = -\{\mathcal{L}_{\lambda_0}p + \mathcal{M}_{\lambda_0}^k q^k + (\lambda - \lambda_0)[(\mathcal{L}_1 - \mathcal{L}_0)u + (\mathcal{M}_1^k - \mathcal{M}_0^k)v^k] + F\}dt + v^k dW_t^k,$$

with the boundary conditions $p|_{t=T} = \phi$ and $p|_{x \in \partial\mathcal{D}} = 0$, has a unique strong solution (p, q) such that $(p, q) \in \mathfrak{H}^{2,1}(\mathcal{D})$. Then we define the operator

$$A : \mathcal{H}^{2,1}(\mathcal{D}) \rightarrow \mathcal{H}^{2,1}(\mathcal{D})$$

as follows:

$$A(u, v) = (p, q).$$

Note that $A(u, v) \in \mathfrak{H}^{2,1}(\mathcal{D})$. Then from estimate (4.26), we can easily obtain that for any $(u_i, v_i) \in \mathcal{H}^{2,1}(\mathcal{D})$, $i = 1, 2$,

$$\|A(u_1 - u_2, v_1 - v_2)\|_{\mathcal{H}^{2,1}(\mathcal{D})}^2 \leq \|A(u_1 - u_2, v_1 - v_2)\|_{\mathfrak{H}^{2,1}(\mathcal{D})}^2 \quad (4.28)$$

$$\leq C|\lambda - \lambda_0| \|(u_1 - u_2, v_1 - v_2)\|_{\mathcal{H}^{2,1}(\mathcal{D})}^2. \quad (4.29)$$

Recall that the constant C in (4.28) is independent of λ . Set $\theta = (2C)^{-1}$. Then the operator is contraction in $\mathcal{H}^{2,1}(\mathcal{D})$ as long as $|\lambda - \lambda_0| \leq \theta$, which implies that equation (4.27) is solvable if $|\lambda - \lambda_0| \leq \theta$.

Equation (4.27) is solvable for $\lambda = 0$ in view of Proposition 4.6. Starting from $\lambda = 0$, we get to $\lambda = 1$ in finite steps, and this finishes the proof of solvability of equation (1.1). The proof of Theorem 3.1 is complete. \square

5 Some applications

5.1 A comparison theorem

The comparison theorem plays an essential role in the theory of PDEs and BSDEs. Ma and Yong [15] gives some comparison theorems for strong solutions to the Cauchy problem of degenerate BSPDEs by Itô's formula, which are improved by Du and Meng [5] under the super-parabolicity condition. In this subsection, we prove the following comparison theorem for the strong solution to BSPDE (1.1) and (1.3) in the general C^2 domain.

Theorem 5.1. *Let the conditions of Theorem 3.1 be satisfied, and (p, q) be the unique strong solution to BSPDE (1.1) and (1.3). Suppose for any (ω, t) , $F(t, \cdot) \geq 0$ and $\phi \geq 0$. Then $p(t, \cdot) \geq 0$ a.s., for every $t \in [0, T]$.*

The proof of Theorem 5.1 needs the following lemma. In what follows, we denote $a^- = -(a \wedge 0)$ for $a \in \mathbb{R}$.

Lemma 5.2. *Let the conditions of Theorem 3.1 be satisfied. In addition, assume that a_x and σ_x are bounded (by a constant L). Let (p, q) be the strong solution of equation (1.2). Then for some constant C ,*

$$E \int_{\mathcal{D}} [p(t, x)^-]^2 dx \leq e^{C(T-t)} \left\{ E \int_{\mathcal{D}} [\phi(x)^-]^2 dx + E \int_t^T \int_{\mathcal{D}} [F(s, x)^-]^2 dx ds \right\}. \quad (5.1)$$

Proof. Define the function $h : \mathbb{R} \rightarrow [0, \infty)$ as follows:

$$h(r) = \begin{cases} r^2, & r \leq -1, \\ (6r^3 + 8r^4 + 3r^5)^2, & -1 \leq r \leq 0, \\ 0, & r \geq 0. \end{cases}$$

Then h is C^2 and

$$h(0) = h'(0) = h''(0) = 0, \quad h(-1) = 1, \quad h'(-1) = -2, \quad h''(-1) = 2.$$

For any $\varepsilon > 0$, define $h_\varepsilon(r) = \varepsilon^2 h(r/\varepsilon)$. The function h_ε has the following properties:

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} h_\varepsilon(r) &= (r^-)^2, \quad \lim_{\varepsilon \rightarrow 0} h'_\varepsilon(r) = -2r^-, \quad \text{uniformly;} \\ |h''_\varepsilon(r)| &\leq C, \quad \forall \varepsilon > 0, r \in \mathbb{R}; \quad \lim_{\varepsilon \rightarrow 0} h''_\varepsilon(r) = \begin{cases} 2, & r < 0, \\ 0, & r > 0. \end{cases} \end{aligned}$$

Since a_x and σ_x exists and they are bounded, equation (1.1) can be written into the divergence form. Then applying Itô's formula for Hilbert-valued semi-martingales (see e.g. [4, Page 105]) to $h_\varepsilon(p(t, \cdot))$, and from Green's formula, we obtain that

$$\begin{aligned} & E \int_{\mathcal{D}} h_\varepsilon(\phi(x)) dx - E \int_{\mathcal{D}} h_\varepsilon(p(t, x)) dx \\ &= E \int_t^T \int_{\mathcal{D}} \left\{ -h'_\varepsilon(p) D_i(a^{ij} D_j p + \sigma^{ik} q^k) - h'_\varepsilon(p) [(b^i - D_j a^{ij}) D_i p \right. \\ &\quad \left. - cp + (\nu^k - D_i \sigma^{ik}) q^k + F] + \frac{1}{2} h''_\varepsilon(p) |q|^2 \right\} dx dt \\ &= E \int_t^T \int_{\mathcal{D}} \left\{ \frac{1}{2} h''_\varepsilon(p) (2a^{ij} D_i p D_j p + 2\sigma^{ik} q^k D_i p + |q|^2) \right. \\ &\quad \left. - h'_\varepsilon(p) [(b^i - D_j a^{ij}) D_i p - cp + (\nu^k - D_i \sigma^{ik}) q^k + F] \right\} dx dt. \end{aligned}$$

Setting $\varepsilon \rightarrow 0$, by Lebesgue's Dominated Convergence Theorem, we have

$$\begin{aligned} & E \int_{\mathcal{D}} [\phi(x)^-]^2 dx - E \int_{\mathcal{D}} [(p(t, x)^-)]^2 dx \\ &= E \int_t^T \int_{\mathcal{D}} 1_{\{p \leq 0\}} \cdot \left\{ (2a^{ij} D_i p D_j p + 2\sigma^{ik} q^k D_i p + |q|^2) \right. \\ &\quad \left. - 2p[(b^i - D_j a^{ij}) D_i p - cp + (\nu^k - D_i \sigma^{ik}) q^k + F] \right\} dx dt. \end{aligned}$$

For positive numbers δ, δ_1 , we have

$$\begin{aligned} 2a^{ij} D_i p D_j p + 2\sigma^{ik} q^k D_i p + |q|^2 &\geq 2a^{ij} D_i p D_j p - (1 + \delta) |\sigma^i D_i p|^2 + \frac{\delta}{1 + \delta} |q|^2 \\ &\geq [-2\delta K + (1 + \delta) \kappa] |Dp|^2 + \frac{\delta}{1 + \delta} |q|^2, \\ -p[(b^i - D_j a^{ij}) D_i p - cp + (\nu^k - D_i \sigma^{ik}) q^k] &\geq -\delta_1 (|Dp|^2 + |q|^2) - C(K, L) \delta_1^{-1} |p|^2. \end{aligned}$$

Taking δ and δ_1 small enough (such that $\delta_1 = \min\{-2\delta K + (1 + \delta)\kappa, \frac{\delta}{1+\delta}\} > 0$), we have

$$\begin{aligned}
& E \int_{\mathcal{D}} [\phi(x)^-]^2 dx - E \int_{\mathcal{D}} [(p(t, x)^-)]^2 dx \\
& \geq E \int_t^T \int_{\mathcal{D}} 1_{\{p \leq 0\}} \cdot [-C(\kappa, K, L)|p|^2 - 2pF] dx dt \\
& \geq E \int_t^T \int_{\mathcal{D}} [-C(\kappa, K, L)|p^-|^2 - 2p^-F^-] dx dt \\
& \geq E \int_t^T \int_{\mathcal{D}} [-C(\kappa, K, L)|p^-|^2 - |F^-|^2] dx dt,
\end{aligned}$$

and this along with the Gronwall inequality implies the desired inequality (5.1). \square

Proof of Theorem 5.1. Due to Assumption 3.3, we can construct (e.g., by the standard technique of mollification) two sequences a_n and σ_n with bounded first derivatives in x , which converge uniformly (w.r.t. (ω, t, x)) to a and σ , respectively, with a_n, σ_n satisfying the same assumptions as a, σ (with κ^2 instead of κ). Then, in view of Theorem 3.1, the following BSPDE (for each n)

$$\begin{cases} dp_n = -(a_n^{ij} D_{ij} p_n + b^i D_i p_n - c p_n + \sigma_n^{ik} D_i q_n^k + \nu^k q_n^k + F) dt + q_n^k dW_t^k, \\ p_n|_{x \in \partial \mathcal{D}} = 0, \quad p_n|_{t=T} = \phi \end{cases}$$

has a unique strong solution $(p_n, q_n) \in \mathbb{H}^2 \times \mathbb{H}^1$, such that

$$\| \| p_n \| \|_2^2 + \| \| q_n \| \|_1^2 + E \sup_{0 \leq t \leq T} \| p_n(t, \cdot) \|_1^2 \leq C (\| \| F \| \|_0^2 + E \| \phi \|_1^2), \quad (5.2)$$

where the constant C only depends on K, ρ_0, κ, T and the function γ , and does not depend on n . It is easy to check that the function pair $(p - p_n, q - q_n)$ satisfies the following BSPDE

$$\begin{cases} du = -(a^{ij} D_{ij} u + b^i D_i u - c u + \sigma^{ik} D_i v^k + \nu^k v^k + F_n) dt + v^k dW_t^k, \\ u|_{x \in \partial \mathcal{D}} = 0, \quad u|_{t=T} = 0, \end{cases}$$

with u and v being the unknown functions, where

$$F_n = (a^{ij} - a_n^{ij}) D_{ij} p_n + (\sigma^{ik} - \sigma_n^{ik}) D_i q_n^k.$$

In view of (5.2) and keeping in mind the uniform convergence of a_n and σ_n , we have

$$\| \| F_n \| \|_0 \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

and this along with estimate (3.3) implies that for every $t \in [0, T]$,

$$p_n(t, \cdot) \rightarrow p(t, \cdot), \quad \text{strongly in } L^2(\Omega \times \mathcal{D}).$$

On the other hand, it follows from Lemma 5.2 that $p_n(t, \cdot) \geq 0$ a.s. for every $t \in [0, T]$. Hence we get $p(t, \cdot) \geq 0$ a.s. for every $t \in [0, T]$. The proof is complete. \square

5.2 Semi-linear equations in C^2 domains

Consider the following BSPDE:

$$\begin{cases} dp(t, x) = - [a^{ij}(t, x)D_{ij}p(t, x) + \sigma^{ik}(t, x)D_i q^k(t, x) + F(t, x, p(t, x), q(t, x))]dt \\ \quad + q^k(t, x)dW_t^k, \quad (t, x) \in [0, T] \times \mathcal{D}; \\ p(t, x) = 0, \quad t \in [0, T], \quad x \in \partial\mathcal{D}; \\ p(T, x) = \phi(x), \quad x \in \mathcal{D}. \end{cases} \quad (5.3)$$

Such a BSPDE is associated to a FBSDE in a similar way as shown in Tang [21].

Assumption 5.1. For any $(u, v) \in (H_0^1(\mathcal{D}) \cap H^2(\mathcal{D})) \times H^1(\mathcal{D}; \mathbb{R}^{d_1})$, the function $F(t, x, u, v)$ is predictable as a function taking values in $H^0(\mathcal{D})$. The function F is continuous in (u, v) . Moreover, for any $\varepsilon > 0$, there exists a constant K_ε such that, for any $(u_i, v_i) \in (H_0^1(\mathcal{D}) \cap H^2(\mathcal{D})) \times H^1(\mathcal{D}; \mathbb{R}^{d_1})$, $i = 1, 2$ and any (ω, t) , we have

$$\begin{aligned} \|F(t, \cdot, u_1, v_1) - F(t, \cdot, u_2, v_2)\|_{0,\mathcal{D}} &\leq \varepsilon(\|u_1 - u_2\|_{2,\mathcal{D}} + \|v_1 - v_2\|_{1,\mathcal{D}}) \\ &\quad + K_\varepsilon(\|u_1 - u_2\|_{0,\mathcal{D}} + \|v_1 - v_2\|_{0,\mathcal{D}}). \end{aligned} \quad (5.4)$$

Remark 5.1. Take $F(t, x, u, v) = b^i D_i u - cu + \nu^k v^k$, and equation (5.3) becomes BSPDE (1.1) and (1.3). Under the boundedness of b, c and ν , we can easily check condition (5.4) by the interpolation inequality. On the other hand, we shall see that the strong solution of BSPDE (5.3) belongs to $\mathcal{H}_0^{2,1}(\mathcal{D})$ ($\subset \mathcal{H}^{2,1}(\mathcal{D})$), which allows us to discuss equation (1.1) with the coefficients c, ν blowing up near the boundary of \mathcal{D} . However, we prefer not to pursue this issue any further in this paper.

Then we have the following

Theorem 5.3. *Let the functions a and σ satisfy Assumptions 3.2 and 3.3. Let Assumptions 3.1 and 5.1 be satisfied. Suppose*

$$F(\cdot, \cdot, 0, 0) \in \mathbb{H}^0(\mathcal{D}), \quad \phi \in L^2(\Omega, \mathcal{F}_T, H_0^1(\mathcal{D})). \quad (5.5)$$

Then BSPDE (5.3) has a unique strong solution $(p, q) \in \mathfrak{H}^{2,1}(\mathcal{D})$ such that

$$\|(p, q)\|_{\mathfrak{H}^{2,1}(\mathcal{D})}^2 \leq C(\|F(\cdot, \cdot, 0, 0)\|_{0,\mathcal{D}}^2 + E\|\phi\|_{1,\mathcal{D}}^2), \quad (5.6)$$

where the constant C only depends on K, ρ_0, κ, T , the functions γ and K_ε .

Proof. Step 1. First, we prove the a priori estimate (5.6) for the strong solution of equation (5.3).

From estimate (3.3), there exists a constant C depending only K, ρ_0, κ, T , and the function γ , such that

$$\|(p, q)\|_{\mathfrak{H}^{2,1}(\mathcal{D})}^2 \leq C(\|F(\cdot, \cdot, p, q)\|_{0,\mathcal{D}}^2 + E\|\phi\|_{1,\mathcal{D}}^2).$$

On the other hand, in view of Assumption 5.1, we have

$$\begin{aligned} \|F(\cdot, \cdot, p, q)\|_{0,\mathcal{D}}^2 &\leq 2\|F(\cdot, \cdot, p, q) - F(\cdot, \cdot, 0, 0)\|_{0,\mathcal{D}}^2 + 2\|F(\cdot, \cdot, 0, 0)\|_{0,\mathcal{D}}^2 \\ &\leq 4\varepsilon^2(\|p\|_{2,\mathcal{D}}^2 + \|q\|_{1,\mathcal{D}}^2) + 4K_\varepsilon^2(\|p\|_{0,\mathcal{D}}^2 + \|q\|_{0,\mathcal{D}}^2) + 2\|F(\cdot, \cdot, 0, 0)\|_{0,\mathcal{D}}^2. \end{aligned} \quad (5.7)$$

Therefore, taking $\varepsilon = (8C)^{-2}$, we have

$$\|(p, q)\|_{\mathfrak{H}^{2,1}(\mathcal{D})}^2 \leq C \left(\|F(\cdot, \cdot, 0, 0)\|_{0,\mathcal{D}}^2 + E\|\phi\|_{1,\mathcal{D}}^2 + \|p\|_{0,\mathcal{D}}^2 + \|q\|_{0,\mathcal{D}}^2 \right). \quad (5.8)$$

Proceeding similarly as in the proof of Proposition 4.6, we can remove the last two terms in the last inequality. Indeed, applying Itô's formula for Hilbert-valued semi-martingales to $[p(t, \cdot)]^2$, we have

$$\begin{aligned} \|p(0, \cdot)\|_{0,\mathcal{D}}^2 &= \|\phi\|_{0,\mathcal{D}}^2 + 2 \int_0^T \int_{\mathcal{D}} p(a^{ij} D_{ij} p + \sigma^{ik} D_i q^k + F(t, x, p, q)) dx dt \\ &\quad - \int_0^T \|q(t, \cdot)\|_{0,\mathcal{D}}^2 dt - 2 \int_0^T \int_{\mathcal{D}} p q^k dx dW_t^k. \end{aligned}$$

Taking expectations, using the Cauchy-Schwarz inequality, and keeping (5.7) in mind, we have for any $\delta > 0$

$$\begin{aligned} \|q\|_{0,\mathcal{D}}^2 &\leq E\|\phi\|_{0,\mathcal{D}}^2 + 2E \int_0^T \int_{\mathcal{D}} p(a^{ij} D_{ij} p + \sigma^{ik} D_i q^k + F(t, x, p, q)) dx dt \\ &\leq E\|\phi\|_{0,\mathcal{D}}^2 + \delta \left(\|p\|_{2,\mathcal{D}}^2 + \|q\|_{1,\mathcal{D}}^2 + \|F(\cdot, \cdot, p, q)\|_{0,\mathcal{D}}^2 \right) + C(\delta, K) \|p\|_{0,\mathcal{D}}^2 \\ &\leq E\|\phi\|_{0,\mathcal{D}}^2 + \delta \left(\|p\|_{2,\mathcal{D}}^2 + \|q\|_{1,\mathcal{D}}^2 \right) + C(\delta, K) \left(\|p\|_{0,\mathcal{D}}^2 + \|F(\cdot, \cdot, 0, 0)\|_{0,\mathcal{D}}^2 \right). \end{aligned}$$

Taking δ small enough and repeating (5.8), we have

$$\|(p, q)\|_{\mathfrak{H}^{2,1}(\mathcal{D})}^2 \leq C \left(\|F(\cdot, \cdot, 0, 0)\|_{0,\mathcal{D}}^2 + E\|\phi\|_{1,\mathcal{D}}^2 + \|p\|_{0,\mathcal{D}}^2 \right),$$

where the constant C only depends on K, ρ_0, κ, T , and the functions γ and K_ε . Using the Gronwall inequality, we obtain a priori estimate (5.6).

Furthermore, from a similar argument as above, we can prove the uniqueness of the strong solution of BSPDE (5.3).

Step 2. We use the method of continuation to prove the solvability of BSPDE (5.3). For each $\lambda \in [0, 1]$, we consider the equation

$$dp = -[a^{ij} D_{ij} p + \sigma^{ik} D_i q + \lambda F(t, x, p, q)] dt + q^k dW_t^k, \quad p|_{x \in \partial \mathcal{D}} = 0, \quad p|_{t=T} = \phi. \quad (5.9)$$

It is clear that the function λF satisfies Assumption 5.1 with the same K_ε as F , and then equation (5.9) has a priori estimate (5.6) for each λ with the same constant C .

Assume that for $\lambda = \lambda_0 \in [0, 1]$, BSPDE (5.9) has a unique strong solution $(p, q) \in \mathfrak{H}^{2,1}(\mathcal{D})$, for any $\phi \in L^2(\Omega, \mathcal{F}_T, H_0^1(\mathcal{D}))$ and any $F \in \mathbb{H}^0(\mathcal{D})$ satisfying Assumption 5.1 and condition (5.5). For other $\lambda \in [0, 1]$, we can rewrite (4.27) as

$$dp = -\{a^{ij} D_{ij} p + \sigma^{ik} D_i q + \lambda_0 F(t, x, p, q) + (\lambda - \lambda_0) F(t, x, p, q)\} dt + q^k dW_t^k.$$

Thus for any $(u, v) \in \mathcal{H}^{2,1}(\mathcal{D})$, the equation

$$dp = -\{a^{ij} D_{ij} p + \sigma^{ik} D_i q + \lambda_0 F(t, x, p, q) + (\lambda - \lambda_0) F(t, x, u, v)\} dt + v^k dW_t^k,$$

with the boundary conditions $p|_{t=T} = \phi$ and $p|_{x \in \partial \mathcal{D}} = 0$, has a unique strong solution $(p, q) \in \mathfrak{H}^{2,1}(\mathcal{D})$. Then define the operator

$$A : \mathcal{H}^{2,1}(\mathcal{D}) \rightarrow \mathcal{H}^{2,1}(\mathcal{D})$$

as $A(u, v) = (p, q)$. Note that $A(u, v) \in \mathfrak{H}^{2,1}(\mathcal{D})$. Proceeding similarly as in Step 1, we can easily obtain that for any $(u_i, v_i) \in \mathcal{H}^{2,1}(\mathcal{D})$, $i = 1, 2$,

$$\|A(u_1 - u_2, v_1 - v_2)\|_{\mathcal{H}^{2,1}(\mathcal{D})}^2 \leq \|A(u_1 - u_2, v_1 - v_2)\|_{\mathfrak{H}^{2,1}(\mathcal{D})}^2 \quad (5.10)$$

$$\leq C|\lambda - \lambda_0| \|(u_1 - u_2, v_1 - v_2)\|_{\mathcal{H}^{2,1}(\mathcal{D})}^2. \quad (5.11)$$

Recall that the constant C in (5.10) does not depend on λ . Set $\theta = (2C)^{-1}$. Then the operator A is a contraction in $\mathcal{H}^{2,1}(\mathcal{D})$ as long as $|\lambda - \lambda_0| \leq \theta$, which implies that (5.9) is solvable if $|\lambda - \lambda_0| \leq \theta$.

The solvability of equation (5.3) for $\lambda = 0$ has been given by Theorem 3.1. Starting from $\lambda = 0$, we can reach $\lambda = 1$ in finite steps, and this finishes the proof of solvability of equation (5.3). \square

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